QFT I - Problem Set 6

(11) Gelfand-Yaglom formalism

We need the results from this exercise for the problem (12) below. Consider the following $N \times N$-matrix

$$\tilde{M}_N = \begin{pmatrix} 2-\frac{\epsilon^2}{m}c_1 & -1 & & & & \\ -1 & 2-\frac{\epsilon^2}{m}c_2 & -1 & & & \\ & -1 & 2-\frac{\epsilon^2}{m}c_3 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2-\frac{\epsilon^2}{m}c_{N-1} & -1 \\ & & & & -1 & 2-\frac{\epsilon^2}{m}c_N \end{pmatrix} .$$

We define the determinant d_N and the subdeteminants d_j as

$$d_N = \det \tilde{M}_N \,, \quad d_j = \det \left(\tilde{M}_N\right)_{j\times j} ,$$

where $\left(\tilde{M}_N\right)_{j\times j}$ is the matrix of the first j rows and columns of $\tilde{M}_N$, $d_0 \equiv 1$ (definition).

a) Show the recursion relation

$$d_{j+1} = \left(2 - \frac{\epsilon^2}{m}\, c_{j+1}\right) d_j - d_{j-1} \,.$$

b) Use the notation $d(t_j) \equiv d(t_{initial} + j\epsilon) = d_j$ and show that in the limit $\epsilon \to 0$, $d(t)$ satisfies the differential equation

$$\frac{d^2}{dt^2} d(t) = -\frac{c(t)}{m} d(t) \,.$$

In this way, the calculation of the determinant of such a big matrix is traced back to the integration of an ordinary differential equation!

(12) Propagator from the Path Integral in case of Quadratic Actions

This exercise generalizes problem (10). We consider a classical action of the form

$$S[q] = \int dt \left(\frac{1}{2} m\dot{q}^2 - \frac{1}{2}\, c(t) q^2\right) .$$

a) Similar like in problem (10) study

$$\langle q_f, t_f | q_i, t_i\rangle = \int_{q_i(t_i)}^{q_f(t_f)} Dq\, e^{iS[q]} \,.$$

Derive the following representation for the propagator

$$\langle q_f, t_f | q_i, t_i\rangle = \sqrt{\frac{m}{2\pi i D(t_f, t_i)}}\, e^{iS[q_0; q_i, q_f]} ,$$

and find a differential equation for $D(t, t_i)$ (with initial values) – we will solve the differential equation for a special case in b) below.

Hints:
(i) As intermediate result you should find similar to problem (10)

$$\langle 0, t_f | 0, t_i \rangle = \lim_{N\to\infty} \sqrt{\frac{m}{2\pi i dt}} \frac{1}{\sqrt{\det \tilde{M}_N}} \,,$$

with $\tilde{M}_N$ *studied in problem (11)* ($\epsilon = dt$).

(ii) To derive a differential equation for

$$D(t, t_i) = \lim_{N\to\infty, dt\to 0} dt \, \det \tilde{M}_N \,,$$

with initial values $D(t_i, t_i)$ *and* $\dot{D}(t_i, t_i)$*, use the solution to problem (11).*

b) Apply the result of a) to the harmonic oscillator: $c(t) = m\omega^2 = const.$ Solve the differential equation for D and compute $S[q_0; q_i, q_f]$ in order to give an explicit expression for the propagator.

Hint: For the calculation of the classical action $S[q_0; q_i, q_f]$ *of the harmonic oscillator use a classical trajectory with* $q(t_i) = q_i$ *and* $q(t_f) = q_f$.

Comment: In the case $c(t) = 0$*, one finds* $D(t_f, t_i) = t_f - t_i$ *and hence the result of problem (10).*