QFT II - Problem Set 8

(44) Thermal field theory and critical temperature

Consider a scalar ϕ^4 theory. We computed in (39) that the effective potential is

$$U(\rho) = m_0^2\rho + \frac{\lambda}{2}\rho^2 + \frac{1}{2}\int_q \ln\left(q^2 + m_0^2 + 3\lambda\rho\right),$$

and derivative w.r.t ρ

$$U'(\rho) = \frac{\partial U(\rho)}{\partial\rho} = m_0^2 + \lambda\rho + \frac{3}{2}\lambda\int_q \frac{1}{q^2 + m_0^2 + 3\lambda\rho}. \tag{1}$$

In thermal field theory, we substitute

$$\int_q \rightarrow T\sum_n \int \frac{d^3q}{(2\pi)^3}$$

and

$$q^2 = (q^0)^2 + \boldsymbol{q}^2 \rightarrow (2\pi nT)^2 + \boldsymbol{q}^2.$$

a) Use the formula

$$\sum_n \frac{1}{n^2 + \frac{x}{(2\pi)^2}} = \frac{2\pi^2}{\sqrt{x}}\left(1 + \frac{2}{\exp(\sqrt{x}) - 1}\right)$$

to perform the Matsubara sum in the expression for U', Equation (1).

b) Your result so far is

$$U'(T,\rho) = m_0^2 + \lambda\rho + \frac{3}{4}\lambda\int \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\omega_q} + \frac{3}{2}\lambda\int \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\omega_q}\frac{1}{\exp(\beta\omega_q) - 1},$$

where $\omega_q \equiv \sqrt{\boldsymbol{q}^2 + m_0^2 + 3\lambda\rho}$ and $\beta \equiv T^{-1}$. The first integral above is temperature independent, i.e. contributes at $T = 0$.

i) Use a cut-off Λ to regularize this integral, to obtain and expression in leading order of Λ. Use this to define a renormalized mass m_R^2 that absorbs the $T = 0$ contribution, i.e $U'(T,\rho) = m_R^2 + \lambda\rho + Rest(T,\rho)$.

ii) Show that the $T = 0$ integral is indeed the one you get for the usual ($T = 0$) field theory i.e compare to

$$\frac{3}{2}\lambda\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{q^2 + m^2 + 3\lambda\rho}$$

c) Having regularize the mass, we may substitute it in ω_q, i.e $\omega_q = \sqrt{\boldsymbol{q}^2 + m_R^2 + 3\lambda\rho}$. For the T-dependent integral

$$\frac{3}{2}\lambda\int_0^\infty \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\omega_q}\frac{1}{\exp(\beta\omega_q) - 1},$$

i) scale q to make it dimensionless, i.e. $q \rightarrow \tilde{q} = q/T$, and expand the integrand in orders of $(m_R^2 + 3\lambda\rho)/T^2$ to convince yourself that the leading order behaviour in the large T limit, i.e.

$T \gg m_R^2 + 3\lambda\rho$ is $const\lambda T^2$.

ii) At vanishing m_R^2 and in the symmetric phase $\rho = 0$, i.e. $\omega_q = q$ which corresponds to the $T \to \infty$ limit, perform the d^3q integration to get $U'(T, \rho = 0)$ in the neighbourhood of $m_R^2 = 0$. For this, you will most probably need a table of integrals or a computer algebra system.

d) As you know, a phase transition from the symmetric $\rho = 0$ to the broken phase will occur when $U'(T, \rho = 0) = m^2(T) \to 0$ signaling an ever growing interaction range $R \propto m^{-1}$. In other words, the phase transition is a long range infra red phenomenon. From your expression for $U'(T, \rho = 0)$, what is the critical temperature T_c defined as $m^2(T_c, \rho = 0) = 0$ for which this happens ?

e) The pressure $P(T)$ is given by

$$P(T) = -U(T).$$

Integrate your expression for

$$\frac{\partial U}{\partial \rho} = m_R^2 + \lambda\rho + \frac{3}{2}\lambda \int \frac{\mathrm{d}^3q}{(2\pi)^3} \frac{1}{\omega_q} \frac{1}{\exp(\beta\omega_q) - 1},$$

over ρ to obtain an expression for $U(\rho, T)$. This, you can do by hand using some substitutions etc. But you might also ask a computer or book in case you don't have time.

f) Your result so far is

$$U(T, \rho) = m_R^2\rho + \frac{\lambda}{2}\rho^2 - \frac{1}{\beta} \int \frac{\mathrm{d}q^3}{(2\pi)^3} \ln \frac{\exp(\beta\omega_q)}{\exp(\beta\omega_q) - 1}.$$

Again, at vanishing m_R^2 and ρ, i.e. $\omega_q = q$, perform the d^3q integration to obtain the pressure in the vicinity of $m_R^2 = 0$ up to an additive constant. This constant can be subtracted by subtracting the potential at $T = 0$, i.e.

$$P(T) - P(T = 0) = -U(T) - U(T = 0).$$

h) From your thermodynamics course or your knowledge of particle physics, what result do you expect for photons ?