Quantum field theory in curved spacetime

Assignment 1 – Apr 28

Exercise 1: Quantum fields in an expanding universe

Motivation: In this first exercise, we'll follow how the vacuum state of a scalar evolves in a toy model of an expanding universe. Even though the setup is simple, it already reveals a key feature of quantum fields in curved spacetime: the vacuum isn't as empty as it seems.

Consider a real massive scalar field χ (minimally coupled) in an expanding universe. Its classical action is

$$S = \frac{1}{2}\int \mathrm{d}^4x \left(\chi'^2 - (\partial_i\chi)^2 - m_{\text{eff}}^2\chi^2\right), \tag{1.1}$$

where i denotes spatial indices and prime corresponds to derivative with respect to conformal time. The effective mass m_{eff}^2 is written as

$$m_{\text{eff}}^2 = m^2a^2 - \frac{a''}{a}, \tag{1.2}$$

with a, the scale factor. Assume that m_{eff}^2 is given by

$$m_{\text{eff}}^2(\eta) = \begin{cases} m_0^2, & \eta < 0 \text{ and } \eta > \eta_1, \\ -m_0^2, & 0 < \eta < \eta_1, \end{cases} \tag{1.3}$$

with m_0 a constant.

(**a**) Solve the equations of motion for χ.

(**b**) Construct the early ("in") and late ("out") time vacuum states.

(**c**) Prove that in the "out" region ($\eta > \eta_1$), the state $|0_{\text{in}}\rangle$ (the vacuum in the "in" region) contains particles. In other words, if we initially start in the vacuum the background evolution has created particles.

(**d**) Show that the mean particle number density in a mode $\mathbf{k}$ is given by

$$n_{\mathbf{k}} = \frac{m_0^4}{|k^4 - m_0^4|}\left|\sin\left(\eta_1\sqrt{k^2 - m_0^2}\right)\right|^2. \tag{1.4}$$

Sanity check: What happens in the limit $\eta_1 \to 0$? Why is this the result we expect?

(**e**) Discuss the regimes $k \gg m_0$ and $k \ll m_0$. What is the physical meaning of these limits?

(**a**) We express the scalar field in terms of its spatial Fourier transform

$$\chi_{\mathbf{k}}(t) = \int \mathrm{d}^3x\,\chi e^{i\mathbf{k}\mathbf{x}}, \tag{1.5}$$

which satisfies the equation of motion

$$\chi_{\mathbf{k}}'' + (m_{\text{eff}}^2 + \mathbf{k}^2)\chi_{\mathbf{k}} = 0. \tag{1.6}$$

We solve the equation of motion for the three cases $\eta < 0$, $0 < \eta < \eta_1$, $\eta > \eta_1$ individually, obtaining the mode expansions

$$\chi_{\mathbf{k}} = \begin{cases} a_{\mathbf{k}} e^{-i\omega_{k,+}\eta} + a_{\mathbf{k}}^\dagger e^{i\omega_{k,+}\eta}\,, & \eta < 0\,, \\ b_{\mathbf{k}} e^{-i\omega_{k,-}\eta} + b_{\mathbf{k}}^\dagger e^{i\omega_{k,-}\eta}\,, & 0 < \eta < \eta_1\,, \\ c_{\mathbf{k}} e^{-i\omega_{k,+}\eta} + c_{\mathbf{k}}^\dagger e^{i\omega_{k,+}\eta}\,, & \eta > \eta_1\,, \end{cases} \tag{1.7}$$

with the amplitudes $a_{\mathbf{k}}$, $b_{\mathbf{k}}$ and $c_{\mathbf{k}}$ and the two dispersion relations

$$\omega_{k,\pm} = \sqrt{\mathbf{k}^2 \pm m_0^2}. \tag{1.8}$$

Note that $\omega_{k,-}$ is imaginary for the modes satisfying $m_0^2 > \mathbf{k}^2$. In order to obtain a valid solution, the scalar field has to be continuous and differentiable. This allows us to express the amplitudes $b_{\mathbf{k}}$ and $c_{\mathbf{k}}$ in terms of $a_{\mathbf{k}}$ by requiring continuity and differentiability at $\eta = 0$ and $\eta = \eta_1$. Making the ansatz

$$b_{\mathbf{k}} = \alpha_b a_{\mathbf{k}} + \beta_b a_{\mathbf{k}}^\dagger, \tag{1.9}$$

we obtain the constraints

$$\alpha_b + \beta_b^* = \frac{\omega_{k,-}}{\omega_{k,+}}(-\alpha_b + \beta_b^*) = 1, \tag{1.10}$$

which have the solution

$$\alpha_b = \frac{\omega_{k,+} + \omega_{k,-}}{2\omega_{k,-}}, \qquad \beta_b = \frac{\omega_{k,+} - \omega_{k,-}}{2\omega_{k,-}}. \tag{1.11}$$

Similarly, with the ansatz

$$c_{\mathbf{k}} = \alpha_c b_{\mathbf{k}} + \beta_c b_{\mathbf{k}}^\dagger, \tag{1.12}$$

we obtain the constraints

$$\alpha_c e^{-i\omega_{k,+}\eta_1} + \beta_c^* e^{i\omega_{k,+}\eta_1} = e^{-i\eta_1\omega_{k,-}}, \qquad \frac{\omega_{k,+}}{\omega_{k,-}}(-\alpha_c e^{-i\omega_{k,+}\eta_1} + \beta_c^* e^{i\omega_{k,+}\eta_1}) = e^{-i\omega_{k,-}\eta_1}, \tag{1.13}$$

which have the solution

$$\alpha_c = \frac{\omega_{k,+} + \omega_{k,-}}{\omega_{k,+}} e^{-i(\omega_{k,+} - \omega_{k,-})\eta_1}, \qquad \beta_c = \frac{\omega_{k,+} - \omega_{k,-}}{\omega_{k,+}} e^{i(\omega_{k,+} + \omega_{k,-})\eta_1}. \tag{1.14}$$

Altogether, we find

$$c_{\mathbf{k}} \equiv \alpha_{ac} a_{\mathbf{k}} + \beta_{ac} a_{\mathbf{k}}^\dagger \tag{1.15}$$

$$= \frac{1}{4\omega_{k,+}\omega_{k,-}} \left((\omega_{k,+} + \omega_{k,-})^2 e^{-i(\omega_{k,-} - \omega_{k,+})\eta_1} - (\omega_{k,+} - \omega_{k,-})^2 e^{-i(\omega_{k,-} + \omega_{k,+})\eta_1} \right) a_{\mathbf{k}} \tag{1.16}$$

$$+ \frac{i}{2} \frac{\omega_{,+}^2 - \omega_{,-}^2}{\omega_{k,+}\omega_{k,-}} \sin(\eta_1 \omega_{k,-}) e^{i(\omega_{k,+})\eta_1} a_{\mathbf{k}}^\dagger. \tag{1.17}$$

(**b**) When quantising the scalar, we promote the amplitudes to operators such that

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{k}' - \mathbf{k}), \qquad [c_{\mathbf{k}}, c_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{k}' - \mathbf{k}). \tag{1.18}$$

The vacuum states $|0_{\rm in}\rangle$ and $|0_{\rm out}\rangle$ (for $\eta < 0$ and $\eta > \eta_1$, respectively) vanish when the corresponding annihilation operators act on them, i.e.they satisfy

$$a_{\mathbf{k}}|0_{\rm in}\rangle = 0, \qquad c_{\mathbf{k}}|0_{\rm out}\rangle = 0, \tag{1.19}$$

for all $\mathbf{k}$.

Note: We are working in the Heisenberg picture. So if we start out in the in-vacuum, we also end up in the in-vacuum.

(**c**) To see whether there are particles in the in-vacuum at late times, we act on it with the annihilation operator at late times. If the result is nonzero, there are particles in that state. We obtain

$$c_{\mathbf{k}}|0_{\rm in}\rangle =(\alpha_{ac}a_{\mathbf{k}} + \beta_{ac}a_{\mathbf{k}})^\dagger|0_{\rm in}\rangle, \tag{1.20}$$

$$=\beta_{ac}a^\dagger_{\mathbf{k}}|0_{\rm in}\rangle, \tag{1.21}$$

$$=\frac{i}{2}\frac{\omega_{,+}^2 - \omega_{,-}^2}{\omega_{k,+}\omega_{k,-}}\sin(\eta_1\omega_{k,-})e^{i(\omega_{k,+})\eta_1}a^\dagger_{\mathbf{k}}|0_{\rm in}\rangle, \tag{1.22}$$

where we used Eq. (1.17). This state is clearly non-zero.

(**d**) We define the number-density operator at late times as

$$n_{\rm out} = \frac{c^\dagger_{\mathbf{k}}c_{\mathbf{k}}}{V}, \tag{1.23}$$

with the volume of space V. In the in-vacuum state, the mean number density equals the expectation value of the number-density operator. Thus, we obtain

$$n_{\mathbf{k}} =\langle 0_{\rm in}|n_{\rm out}|0_{\rm in}\rangle \tag{1.24}$$

$$=V^{-1}\langle 0_{\rm in}|c^\dagger_{\mathbf{k}}c_{\mathbf{k}}|0_{\rm in}\rangle, \tag{1.25}$$

$$=V^{-1}||c_{\mathbf{k}}|0_{\rm in}\rangle||^2, \tag{1.26}$$

$$=\frac{(\omega_{k,+}^2 - \omega_{k,-}^2)^2}{4\omega_{k,+}\omega_{k,-}V}|\sin(\omega_{k,-}\eta_1)|^2||a^\dagger_{\mathbf{k}}|0_{\rm in}\rangle||^2, \tag{1.27}$$

$$=\frac{m_0^4}{|\mathbf{k}^4 - m_0^4|}|\sin(\omega_{k,-}\eta_1)|^2\frac{\delta^{(3)}(0)}{V}. \tag{1.28}$$

Oops, $\delta^{(3)}(0)$ incoming. But don't despair! That's just the volume of space that we divide by anyway. This is why, in field theory on unbounded backgrounds, it is just more useful to consider densities. Thus, by a slight of hand $\delta^{(3)}(0)/V = 1$ and we obtain

$$n_{\mathbf{k}} = \frac{m_0^4}{|\mathbf{k}^4 - m_0^4|}|\sin(\omega_{k,-}\eta_1)|^2. \tag{1.29}$$

(**e**) The modes which satisfy $k \gg m_0$ have very large energies and thus probe very small distances. At small distances, spacetime is approximately Minkowskian. In our toy model, the change in the mass is of order m_0, which estimates the scale of background curvature we model. Thus, if $k \gg m_0$ the particles approximately see Minkowski spacetime, and particle creation is negligible. Indeed, we infer from Eq. (1.29) that $\lim_{k\to\infty} n_{\mathbf{k}} = 0$.

If, in turn, $k \ll m_0$ the wave length of the modes is much larger than background-curvature length scales. Therefore, particle creation is maximal. We see this in the number density

$$n_0 = \sinh^4(m_0\eta_1), \tag{1.30}$$

which grows exponentially with $m_0\eta_1$.

Exercise 2: Bogolyubov transformations

Motivation: We've seen that one person's vacuum can be filled with particles from another person's point of view. Now we derive general rules that relate the vacua of different observers.

Given a set of mode functions $v_k(\eta)$ (with conformal time η and $k = |\mathbf{k}|$), a scalar field on a cosmological background can be expanded as

$$\chi = \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}k^3}{(2\pi)^{3/2}} \left(v_k^* a_{\mathbf{k}} e^{i\mathbf{kx}} + v_k a_{\mathbf{k}}^\dagger e^{-i\mathbf{kx}} \right), \tag{2.1}$$

where

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = \delta^{(3)}(\mathbf{k}' - \mathbf{k}). \tag{2.2}$$

Let us define a new set of mode functions as a linear combination

$$u_k = \alpha_k v_k + \beta_k v_k^*. \tag{2.3}$$

The numbers α_k and β_k are called Bogolyubov coefficients. They are related as

$$|\alpha_k|^2 - |\beta_k|^2 = 1. \tag{2.4}$$

Given the new set of mode functions, we can equivalently expand the scalar as

$$\chi = \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}k^3}{(2\pi)^{3/2}} \left(u_k b_{\mathbf{k}} e^{i\mathbf{kx}} + u_k^* b_{\mathbf{k}}^\dagger e^{-i\mathbf{kx}} \right), \tag{2.5}$$

where again

$$[b_{\mathbf{k}}, b_{\mathbf{k}'}^\dagger] = \delta^{(3)}(\mathbf{k}' - \mathbf{k}). \tag{2.6}$$

Then, we can express the different classes of creation and anihilation operators as linear combinations, e.g.

$$b_{\mathbf{k}} = \alpha_k a_{\mathbf{k}} - \beta_k a_{-\mathbf{k}}^\dagger \tag{2.7}$$

In class, you have derived the average particle number density of modes associated with the operator $a^\dagger$ in the b-vacuum. Now, we go a step further and explicitly express the b-vacuum state in terms of creation and anihilation operators of the state a acting on the a-vacuum. The b-vacuum state for a pair of modes $(\mathbf{k}, -\mathbf{k})$ satisfies

$$b_{\mathbf{k}} |0_{\mathbf{k},-\mathbf{k}}^{(b)}\rangle = b_{-\mathbf{k}} |0_{\mathbf{k},-\mathbf{k}}^{(b)}\rangle = 0. \tag{2.8}$$

(a) Expand the b-vacuum in terms of a-particle states.

(b) Use the properties of $|0_{\mathbf{k},-\mathbf{k}}^{(b)}\rangle$ to obtain the expansion coefficients.

(c) Normalize the resulting state. You should obtain the result

$$|0_{\mathbf{k},-\mathbf{k}}^{(b)}\rangle = \frac{1}{|\alpha_k|} e^{\frac{\beta_k}{\alpha_k} a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger} |0_{\mathbf{k},-\mathbf{k}}^{(a)}\rangle. \tag{2.9}$$

(d) Write down the full b-vacuum state in terms of the mode-specific $|0_{\mathbf{k},-\mathbf{k}}^{(b)}\rangle$.

(e) Let's have a closer look at the expansion. What kind of state is the b-vacuum in terms of a-particle states?

(**a**) We can introduce a partition of unity in the Fock space describing modes of wave number $\pm k$ such that

$$\mathbb{1}_{\mathbf{k},-\mathbf{k}} = \frac{1}{|N|}\sum_{n,m=0}^{\infty} |m_{-\mathbf{k}}^{(a)}, n_{\mathbf{k}}^{(a)}\rangle\langle m_{-\mathbf{k}}^{(a)}, n_{\mathbf{k}}^{(a)}|, \tag{2.10}$$

with some normalizing factor $|N|$. Then, the vacuum state of particle b can be expanded as

$$|0_{\mathbf{k},-\mathbf{k}}^{(b)}\rangle = \frac{1}{|N|}\sum_{n,m=0}^{\infty} \langle m_{-\mathbf{k}}^{(a)}, n_{\mathbf{k}}^{(a)}|0_{\mathbf{k},-\mathbf{k}}^{(b)}\rangle|m_{-\mathbf{k}}^{(a)}, n_{\mathbf{k}}^{(a)}\rangle, \tag{2.11}$$

$$= \frac{1}{|N|}\sum_{n,m=0}^{\infty} c_{nm}|m_{-\mathbf{k}}^{(a)} n_{\mathbf{k}}^{(a)}\rangle. \tag{2.12}$$

(**b**) Eq. (2.8) tells us that the vacuum is invariant under parity transformations ($\mathbf{k} \to -\mathbf{k}$). This implies the constraint

$$c_{nm} = c_{mn}. \tag{2.13}$$

Besides, it has to be anihilated by the operator $b_{\mathbf{k}}$ which shifts Using Eq. (2.7), we obtain the relation

$$b_{\mathbf{k}}|0_{\mathbf{k},-\mathbf{k}}^{(b)}\rangle = \frac{1}{|N|}\sum_{n,m=0}^{\infty} c_{nm}(\alpha_k a_{\mathbf{k}} - \beta_k a_{-\mathbf{k}}^{\dagger})|m_{-\mathbf{k}}^{(a)} n_{\mathbf{k}}^{(a)}\rangle \tag{2.14}$$

$$= \frac{1}{|N|}\sum_{n,m=0}^{\infty} c_{nm}(\sqrt{n}\alpha_k|m_{-\mathbf{k}}^{(a)}(n-1)_{\mathbf{k}}^{(a)})\rangle - \sqrt{m+1}\beta_k|(m+1)_{-\mathbf{k}}^{(a)} n_{\mathbf{k}}^{(a)})\rangle \tag{2.15}$$

$$= \frac{1}{|N|}\sum_{n,m=0}^{\infty} (c_{n+1,m}\sqrt{n+1}\alpha_k - c_{n,m-1}\sqrt{m}\beta_k)|m_{-\mathbf{k}}^{(a)} n_{\mathbf{k}}^{(a)}\rangle \tag{2.16}$$

$$= 0. \tag{2.17}$$

Every coefficient of this linear combination has to vanish individually. Thus, we obtain the iterative relations

$$c_{n+1,m}\sqrt{n+1}\alpha_k - c_{n,m-1}\sqrt{m}\beta_k = 0. \tag{2.18}$$

Let's express everything in terms of $c_{0,0}$. Then, immediately $c_{n,0} = c_{0,n} = 0$ for all $n > 0$. But then, again, $c_{n,1} = c_{1,n} = 0$ for all $n > 1$ and so on. Thus, we find that

$$c_{nm} \propto \delta_{mn}. \tag{2.19}$$

For the diagonal elements, we obtain

$$c_{n+1,n+1} = \frac{\beta_k}{\alpha_k} c_{n,n}. \tag{2.20}$$

Thus, we obtain the relation

$$c_{n,n} = \left(\frac{\beta_k}{\alpha_k}\right)^n c_{0,0}. \tag{2.21}$$

Note that from the recurrence relation given in Eq. (2.18) alone one could think that $c_{0,0} = 0$, setting $n = -1$ and $m = 0$. However, $n = -1$ is not contained in the sum in Eq. (2.16) – thus, $n = -1$ is not applicable and $c_{0,0} \neq 0$.

Given our solution to the recurrence relation, we can expand the b-vacuum in the mode $\mathbf{k}$ as

$$|0^{(b)}_{\mathbf{k},-\mathbf{k}}\rangle = \frac{1}{|\bar{N}|}\sum_{n=0}^{\infty}\left(\frac{\beta_k}{\alpha_k}\right)^n |n^{(a)}_{-\mathbf{k}} n^{(a)}_{\mathbf{k}}\rangle \tag{2.22}$$

$$= \frac{1}{|\bar{N}|}\sum_{n=0}^{\infty}\left(\frac{\beta_k}{\alpha_k}\right)^n \frac{(a^\dagger_{\mathbf{k}})^n (a^\dagger_{-\mathbf{k}})^n}{n!}|0^{(a)}_{\mathbf{k},-\mathbf{k}}\rangle \tag{2.23}$$

$$= \frac{1}{|\bar{N}|} e^{\frac{\beta_k}{\alpha_k} a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}}}|0^{(a)}_{\mathbf{k},-\mathbf{k}}\rangle, \tag{2.24}$$

where we absorbed $c_{0,0}$ into $\bar{N}$ and discarded a global phase.

(**c**) The norm of our state reads

$$|||0^{(b)}_{\mathbf{k},-\mathbf{k}}\rangle||^2 = \frac{1}{|\bar{N}|^2}\sum_{n,m}\left(\frac{\beta_k}{\alpha_k}\right)^n \left(\frac{\beta^*_k}{\alpha^*_k}\right)^m \langle m^{(a)}_{-\mathbf{k}} m^{(a)}_{\mathbf{k}}|n^{(a)}_{-\mathbf{k}} n^{(a)}_{\mathbf{k}}\rangle \tag{2.25}$$

$$= \frac{1}{|\bar{N}|^2}\sum_{n}\left(\frac{|\beta_k|^2}{|\alpha_k|^2}\right)^n \tag{2.26}$$

$$= \frac{1}{|\bar{N}|^2}\frac{|\alpha_k|^2}{|\alpha|^2 - |\beta_k|^2} \tag{2.27}$$

$$= \frac{|\alpha_k|^2}{|\bar{N}|^2}. \tag{2.28}$$

Thus, we finally obtain

$$|0^{(b)}_{\mathbf{k},-\mathbf{k}}\rangle = \frac{1}{|\alpha_k|} e^{\frac{\beta_k}{\alpha_k} a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}}}|0^{(a)}_{\mathbf{k},-\mathbf{k}}\rangle. \tag{2.29}$$

Hooray.

(**d**) For different wave numbers, the states live in different Hilbert spaces. Thus, we can just take their tensor product for all values of $\mathbf{k}$

$$|0^{(b)}\rangle = \prod_{\mathbf{k}} \frac{1}{|\alpha_k|} e^{\frac{\beta_k}{\alpha_k} a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}}}|0^{(a)}_{\mathbf{k},-\mathbf{k}}\rangle. \tag{2.30}$$

(**e**) Expressed in terms of a-particle states, the b-vacuum is a two-mode squeezed coherent state. That's a mouthful. Let's break it down:

- That its coherent means that it saturates the uncertainty relation. It is, thus, as classical as it can get. Note that a squeezed coherent state is not necessarily a coherent state (an eigenstate of the anihilation operator), indeed the one we are dealing with is exactly such an example. That's a terminology trap right there!

- That its two-mode squeezed means that it comes in pairs – in this case with opposite wave numbers $(\mathbf{k}, -\mathbf{k})$. More generally, it comes in pairs with exactly opposite quantum numbers, i.e.particle-antiparticle pairs. This has to be the case because quantum numbers have to be conserved.

Exercise 3: Instantaneous vacuum

Motivation: Every mode function allows to construct a different vacuum. What could be a sensible definition of vacuum then? Let's find out!

Ordinarily, we define the vacuum as the lowest-energy state. In cosmology, however, the Hamiltonian is time dependent. Energy is not conserved. This creates particles. Thus, the lowest-energy state at one time (the *instantaneous vacuum*), may not be the lowest-energy state at a different time. Let's see, how this comes about.
As above consider a real massive scalar field, whose dynamics are characterized by the action given in Eq. (1.1). This results in the Hamiltonian

$$H = \frac{1}{2}\int_x \left(\pi^2 + (\partial_i \chi)^2 + m_{\text{eff}}^2 \chi^2\right), \tag{3.1}$$

with the momentum conjugate π. Assume that the field possesses a mode expansion as in Eq. (2.1).

(**a**) Express the Hamiltonian in terms of creation and anihilation operators. You should obtain something of the form

$$H = \frac{1}{4}\int \mathrm{d}^3k \left[a_{\mathbf{k}} a_{-\mathbf{k}} F_{\mathbf{k}}^* + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger F_{\mathbf{k}} + (2a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \delta^{(3)}(0))E_{\mathbf{k}}\right] \tag{3.2}$$

for some $E_{\mathbf{k}}$, $F_{\mathbf{k}}$.

(**b**) Compute the mean energy density in the a-vacuum.

(**c**) Assuming that $\omega_k^2 = k^2 + m_{\text{eff}}^2 > 0$, find initial conditions for the mode function that minimize the mean energy density at conformal time η_0. (**Hint:** Normalize the mode functions.) What is the corresponding Hamiltonian at conformal time η_0? You should obtain that the Hamiltonian is diagonal in this case.

(**d**) Compute the initial conditions for the mode function after an infinitesimal time shift, i.e.at conformal time $\eta_0 + \delta\eta$. Compare these initial conditions to the ones derived in the previous exercise. How do we interpret this result? (**Hint:** Have in mind exercise 2.)

(**e**) Imagine that you could find a vacuum state which diagonalizes the Hamiltonian at all times. Which equation would the mode functions have to satisfy? Is this equation compatible with the equations of motion?

In specific situations, it can happen that the lowest-energy state at one time η_0 amounts to an infinite number density at a different time η_1, even if the geometry changes slowly compared to the time difference that is characteristic of the problem one would like to answer (i.e.adiabatically). This casts serious doubts on the physical interpretation of the instantaneous vacuum.
However, adiabatic evolution allows us to (at least approximately) define a different vacuum state with interesting properties: The adiabatic vacuum. If the energy density is changing slowly during the considered time interval, the equations of motion allow for the approximate solution[a]

$$v_k^{\text{WKB}}(\eta) = \frac{e^{i\int_{\eta_0}^{\eta} \omega_k(\eta')\mathrm{d}\eta'}}{\sqrt{\omega_k}}. \tag{3.3}$$

We can define the adiabatic vacuum $|0_{\text{ad}}(\eta_0)\rangle$ at a time η_0 by finding exact mode functions which satisfy the initial conditions

$$v_k(\eta_0) = v_k^{\text{WKB}}(\eta_0), \qquad v_k'(\eta_0) = v_k^{\text{WKB}\prime}(\eta_0), \tag{3.4}$$

and constructing the vacua relative to the corresponding anihilation operator.

(**f**) Quantify how adiabatic a general background evolution yielding $\omega_k(\eta)$ is. Which condition should an adiabatically evolving background satisfy if the quantum-field evolution is considered in a finite-time interval $\Delta\eta = \eta_1 - \eta_0$?

(**g**) Compute the energy density of the adiabatic vacuum in general. Is it minimal?

[a] This approximation is called WKB (Wentzel–Kramers–Brillouin) approximation, a standard method in quantum mechanics in general.

(**a**) Given the mode expansion in Eq. (2.1), the conjugate momentum reads

$$\pi(x) = \partial_\eta \chi = \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \left(v_k'^* a_{\mathbf{k}} e^{i\mathbf{kx}} + v_k' a_{\mathbf{k}}^\dagger e^{-i\mathbf{kx}} \right). \tag{3.5}$$

Thus, it's square contributes

$$\pi(x)^2 = \frac{1}{2} \int \frac{\mathrm{d}^3 k \mathrm{d}^3 k'}{(2\pi)^3} \left(v_k'^* a_{\mathbf{k}} e^{i\mathbf{kx}} + v_k' a_{\mathbf{k}}^\dagger e^{-i\mathbf{kx}} \right) \left(v_{k'}'^* a_{\mathbf{k}'} e^{i\mathbf{k}'\mathbf{x}} + v_{k'}' a_{\mathbf{k}'}^\dagger e^{-i\mathbf{k}'\mathbf{x}} \right), \tag{3.6}$$

$$= \frac{1}{2} \int \mathrm{d}^3 k \left(v_k'^{*2} a_{\mathbf{k}} a_{-\mathbf{k}} + |v_k'|^2 (a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger a_{\mathbf{k}}) + v_k'^2 a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger \right), \tag{3.7}$$

Similarly, the gradient of the scalar has the square

$$(\nabla\chi)^2 = \frac{1}{2} \int \mathrm{d}^3 k k^2 \left(v_k^{*2} a_{\mathbf{k}} a_{-\mathbf{k}} + |v_k|^2 (a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger a_{\mathbf{k}}) + v_k^2 a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger \right). \tag{3.8}$$

Then, the Hamiltonian reads

$$H = \frac{1}{4} \int \mathrm{d}^3 k \left[(v_k'^{*2} + \omega_k^2 v_k^{*2}) a_{\mathbf{k}} a_{-\mathbf{k}} + (v_k'^2 + \omega_k^2 v_k^2) a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger + (|v_k'|^2 + \omega_k^2 |v_k|^2)) (a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger a_{\mathbf{k}}) \right], \tag{3.9}$$

$$= \frac{1}{4} \int \mathrm{d}^3 k \left[F_{\mathbf{k}}^* a_{\mathbf{k}} a_{-\mathbf{k}} + F_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger + E_{\mathbf{k}} (2 a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \delta^{(3)}(0)) \right], \tag{3.10}$$

where we defined

$$F_{\mathbf{k}} = v_k'^2 + \omega_k^2 v_k^2, \qquad E_{\mathbf{k}} = |v_k'|^2 + \omega_k^2 |v_k|^2. \tag{3.11}$$

(**b**) The only nonvanishing contribution to the expectation value of the Hamiltonian in the a-vacuum reads

$$\langle 0^{(a)} | H | 0^{(a)} \rangle = \frac{\delta^{(3)}(0)}{4} \int \mathrm{d}^3 k E_{\mathbf{k}}. \tag{3.12}$$

The factor $\delta^{(3)}(0)$ amounts to the infinite volume of space. Therefore, we define the mean energy density as

$$\rho \equiv \frac{\langle 0^{(a)} | H | 0^{(a)} \rangle}{\delta^{(3)}(0)} = \frac{1}{4} \int \mathrm{d}^3 k E_{\mathbf{k}}. \tag{3.13}$$

(**c**) We have to minimize the energy density for each mode individually. Thus, we should minimize $E_{\mathbf{k}}(\eta_0)$. Note that the normalization condition for the mode functions reads

$$v_k' v_k^* - v_k'^* v_k = 2i. \tag{3.14}$$

We can shift the phase of the mode function by a constant $v_k \to e^{i\lambda} v_k$ without changing the physics. Such a shift implies $v_k' \to e^{i\lambda} v_k'$. By such a shift, we can make $v_k(\eta_0)$ real. Thus, the normalization condition, Eq. (3.14), at time η_0 reads

$$v_k(\eta_0) = \frac{1}{\mathrm{Im}(v_k')(\eta_0)}. \tag{3.15}$$

Thus, we have to minimize the quantity

$$E_{\mathbf{k}}(\eta_0) = [\mathrm{Re}(v_k')(\eta_0)]^2 + [\mathrm{Im}(v_k')(\eta_0)]^2 + \frac{\omega_k^2(\eta_0)}{[\mathrm{Im}(v_k')(\eta_0)]^2}, \tag{3.16}$$

with respect to both $x = \mathrm{Re}(v_k')(\eta_0)$ and $y = \mathrm{Im}(v_k')(\eta_0)$ individually. Thus, we have to minimize the functions x^2, yielding $x = 0$, and $y^2 + \omega_k^2(\eta_0)/y^2$, yielding $y = \sqrt{\omega_k(\eta_0)}$. We obtain the solution

$$v_k(\eta_0) = \frac{1}{\sqrt{\omega_k(\eta_0)}}, \qquad v_k'(\eta_0) = i\sqrt{\omega_k(\eta_0)} = i\omega_k(\eta_0)v_k(\eta_0). \tag{3.17}$$

Here, we had to assume that that $\omega_k^2(\eta_0) > 0$ – otherwise the manifestly real quantity $\mathrm{Im}(v_k')(\eta_0)$ would have been imaginary. This was expected, because the energy of modes with negative ω_k^2 is unbounded from below, thus not allowing for a minimal energy density.

Eq. (3.17) implies that $F_{\mathbf{k}}(\eta_0) = 0$, while

$$E_{\mathbf{k}}(\eta_0) = 2\omega_k(\eta_0). \tag{3.18}$$

As a result, the Hamiltonian at conformal time η_0 is diagonal (i.e.a function of the number operator $N_{\mathbf{k}} \equiv a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$ only), reading

$$H(\eta_0) = \int \mathrm{d}^3k\omega_k(\eta_0)\left[N_{\mathbf{k}} + \frac{\delta^{(3)}(0)}{2}\right]. \tag{3.19}$$

Note: The corresponding vacuum energy density

$$\rho(\eta_0) = \frac{1}{2}\int \mathrm{d}^3k\omega_k(\eta_0), \tag{3.20}$$

is divergent, and has to be renormalized. Simply subtracting it away, as one usually does it in flat space, does not work because it is time dependent.

(**d**) The initial conditions for the mode functions defining the instantaneous vacuum at conformal time $\eta_0 + \delta\eta$ read

$$u_k(\eta_0 + \delta\eta) = \frac{1}{\sqrt{\omega_k(\eta_0 + \delta\eta)}}, \qquad u_k'(\eta_0 + \delta\eta) = i\sqrt{\omega_k(\eta_0 + \delta\eta)}. \tag{3.21}$$

Expanding in the infinitesimal $\delta\eta$,

$$u_k(\eta_0) + u_k'(\eta_0)\delta\eta \simeq \frac{1}{\sqrt{\omega_k(\eta_0)}} - \frac{1}{2}\frac{\omega_k'(\eta_0)\delta\eta}{\omega_k^{3/2}(\eta_0)}, \quad u_k'(\eta_0) + u_k''(\eta_0)\delta\eta = i\left(\sqrt{\omega_k(\eta_0)} + \frac{1}{2}\frac{\omega_k'(\eta_0)\delta\eta}{\sqrt{\omega_k(\eta_0)}}\right). \tag{3.22}$$

Thus, the mode functions u and v can only be equal if

$$\omega_k'(\eta_0) = -2i\omega_k^2(\eta_0), \qquad u_k''(\eta_0) = \frac{1}{2}\frac{\omega_k'(\eta_0)}{\sqrt{\omega_k(\eta_0)}}. \tag{3.23}$$

The first of these two equalities provides an initial condition for the background at η_0. The second equality is an additional initial condition for the mode, which together with the equation of motion yields

$$u_k''(\eta_0) + \omega_k^2(\eta_0)u_k(\eta_0) = 2\left(\frac{i}{2}\omega_k'(\eta_0)\right)^{3/4} = 0. \tag{3.24}$$

Thus, u and v are equal iff $\omega_k'(\eta_0) \propto m_{\text{eff}}^{2\prime}(\eta_0) = 0$, i.e.the background should not evolve at time η_0.

Thus, using the language from exercise 2, the operators $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ are related by a nontrivial Bogolyubov transformation, and the a-vacuum is filled with b-particles. In other words, even after an infinitesimal shift in conformal time, the state of minimal energy density is not the state of minimal energy density any more, unless the background is static during that time.

(**e**) In order to diagonalize the Hamiltonian, the mode function has to satisfy the differential equation

$$F_{\mathbf{k}} = v_k'^2 + \omega_k^2 v_k^2 = 0. \tag{3.25}$$

Taking a derivative with respect to conformal time, we obtain

$$v_k'' = i(\omega_k' v_k + \omega_k v_k') = (i\omega_k' - \omega_k^2) v_k. \tag{3.26}$$

At the same time, the equations of motion require that

$$v_k'' = -\omega_k^2 v_k. \tag{3.27}$$

Thus, the Hamiltonian can only be diagonalized by one mode function in a way consistent with the equations of motion if

$$\omega_k' = 0, \tag{3.28}$$

i.e.in a non-evolving background. In other words, in a cosmological setting it is impossible to diagonalize the Hamiltonian for all times.

(**f**) Being an energy, $\omega_k(\eta)$ has an associated characteristic frequency and with that an associated characteristic period

$$\Delta\eta = \frac{2\pi}{\omega_k(\eta)}. \tag{3.29}$$

For ω_k to be slowly changing, it then has to satisfy

$$\left| \frac{\omega(\eta + \Delta\eta) - \omega(\eta)}{\omega(\eta)} \right| \ll 1. \tag{3.30}$$

Expanding in $\Delta\eta$ (which we should be allowed to do if ω_k is slowly changing), we obtain

$$\left| \frac{\omega_k'(\eta)\Delta\eta}{\omega_k(\eta)} \right| = 2\pi \left| \frac{\omega_k'(\eta)}{\omega_k^2(\eta)} \right| \ll 1. \tag{3.31}$$

(**g**) The initial conditions for the mode functions defining the adiabatic vacuum (derived from Eq. (3.4)) are

$$v_k(\eta_0) = \frac{1}{\sqrt{\omega_k(\eta_0)}}, \qquad v_k'(\eta_0) = i\sqrt{\omega_k(\eta_0)} \left[1 + \frac{i}{2} \frac{\omega_k'}{\omega_k^2} \right]. \tag{3.32}$$

These deviate from the minimal-energy-density initial conditions by a term proportional to ω_k'/ω_k^2, which as we derived in the previous exercise is small for slowly changing backgrounds. Thus, we expect to be close to the minimal energy density. Indeed, we obtain

$$\rho = \frac{1}{4} \int \mathrm{d}^3 k E_k = \frac{1}{2} \int \mathrm{d}^3 k \omega_k \left(1 + \frac{1}{16} \left| \frac{\omega_k'}{\omega_k^2} \right|^2 \right). \tag{3.33}$$

Considering that the corrections are quadratic in a very small number, this is indeed very close to minimal energy density.