Quantum field theory in curved spacetime

Assignment 2 – May 5

Exercise 4: Conformally coupled scalar field

Motivation: In the lecture, we saw that nonminimally coupled scalar fields are not produced in an FLRW background if they are conformally coupled. But why those specific coupling values? Here, we'll work out precisely what conformal coupling means.

Consider a scalar field ϕ non-minimally coupled with gravity. It is described by the following action (**Note that in the original sheet there was a minus sign missing in front of the kinetic term of the scalar**)

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{m^2}{2} \phi^2 - \frac{\xi}{2} R \phi^2 \right) . \tag{4.1}$$

(a) Consider the conformal transformation

$$g_{\mu\nu}(x,t) \to \Omega^2(x,t) g_{\mu\nu}(x,t), \tag{4.2}$$
$$\phi(x,t) \to \Omega^{-1}(x,t)\phi(x,t). \tag{4.3}$$

Calculate the value of ξ under which the action is invariant under this conformal transformation (possibly up to a total divergence).

(b) Show that the energy-momentum tensor $T^{\mu\nu}$ is expressed as

$$\begin{aligned} T^{\mu\nu} &= \nabla^\mu \phi \nabla^\nu \phi - \frac{1}{2} g^{\mu\nu} \nabla^\rho \phi \nabla_\rho \phi + \frac{1}{2} g^{\mu\nu} m^2 \phi^2 - \xi \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \phi^2 \\ &+ \xi \left[g^{\mu\nu} \nabla^\alpha \nabla_\alpha (\phi^2) - \nabla^\mu \nabla^\nu (\phi^2) \right] . \end{aligned} \tag{4.4}$$

(c) Show that $T^\nu{}_\nu = 0$ when $m = 0$ and $\xi = 1/6$.

(a) From here on on barred quantities denote the transformed metric such that $\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$. The inverse metric transforms as $\bar{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}$ (recall that $\bar{g}^{\mu\nu} \bar{g}_{\nu\rho} = \delta^\mu_\rho$). As a result, the Christoffel symbol transforms as

$$\bar{\Gamma}^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\mu\nu} + \frac{g^{\rho\sigma}}{\Omega} \left(2 g_{\sigma(\nu} \nabla_{\mu)} \Omega - g_{\mu\nu} \nabla_\sigma \Omega \right) , \tag{4.5}$$

where we use the notation

$$g_{\rho(\nu} \nabla_{\mu)} = \frac{1}{2} \left(g_{\rho\nu} \nabla_\mu + g_{\rho\mu} \nabla_\nu \right) . \tag{4.6}$$

Then, the Ricci tensor reads

$$\bar{R}_{\mu\nu} = \partial_\rho \bar{\Gamma}^\rho{}_{\mu\nu} - \partial_\nu \bar{\Gamma}^\rho{}_{\rho\mu} + \bar{\Gamma}^\rho{}_{\rho\sigma} \bar{\Gamma}^\sigma_{\mu\nu} - \bar{\Gamma}^\rho{}_{\mu\sigma} \bar{\Gamma}^\sigma_{\nu\rho}, \tag{4.7}$$

$$= R_{\mu\nu} - 2 \frac{\nabla_\mu \nabla_\nu \Omega}{\Omega} - g_{\mu\nu} \frac{\Box \Omega}{\Omega} + 4 \frac{\nabla_\mu \Omega \nabla_\nu \Omega}{\Omega^2} - g_{\mu\nu} \frac{g^{\rho\sigma} \nabla_\rho \Omega \nabla_\sigma \Omega}{\Omega^2}, \tag{4.8}$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. Finally, the Ricci scalar reads

$$\bar{R} = \frac{R}{\Omega^2} - 6\frac{\Box\Omega}{\Omega^3}. \tag{4.9}$$

At the same time, the determinant of the metric transforms as

$$\bar{g} \equiv \epsilon^{\mu_1\cdots\mu_4}\bar{g}_{1\mu_1}\dots\bar{g}_{4\mu_4} = \Omega^8 g. \tag{4.10}$$

Together with the transformation of the scalar, we obtain

$$S = \int \mathrm{d}^4x\sqrt{-\bar{g}}\left(-\frac{1}{2}\bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\phi}\bar{\nabla}_\nu\bar{\phi} - \frac{m^2}{2}\bar{\phi}^2 - \frac{\xi}{2}\bar{R}\bar{\phi}^2\right), \tag{4.11}$$

$$= \int \mathrm{d}^4x\sqrt{g}\left[-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + \phi\nabla_\mu\phi g^{\mu\nu}\frac{\nabla_\nu\Omega}{\Omega} - \frac{1}{2}\phi^2 g^{\mu\nu}\frac{\nabla_\mu\Omega\nabla_\nu\Omega}{\Omega^2} - \frac{m^2}{2}\phi^2\Omega^2 - \frac{\xi}{2}\phi^2\left(R - 6\frac{\Box\Omega}{\Omega}\right)\right], \tag{4.12}$$

$$= \int \mathrm{d}^4x\sqrt{g}\left[-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{m^2}{2}\phi^2\Omega^2 - \frac{\xi}{2}\phi^2 R + 3\left(\xi - \frac{1}{6}\right)\phi^2\frac{\Box\Omega}{\Omega}\right], \tag{4.13}$$

where we used partial integration to obtain the equality

$$\int \mathrm{d}^4x\sqrt{-g}\frac{\phi^2}{2}g^{\mu\nu}\frac{\nabla_\mu\Omega\nabla_\nu\Omega}{\Omega^2} = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{\phi^2}{2}\frac{\Box\Omega}{\Omega} + \phi\nabla_\mu\phi g^{\mu\nu}\frac{\nabla_\nu\Omega}{\Omega}\right]. \tag{4.14}$$

Thus, the action is invariant if $m = 0$ and $\xi = 1/6$.

(**b**) The energy-momentum tensor is defined as

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}. \tag{4.15}$$

We, thus, have to vary the action with respect to the inverse metric. Let us start with the metric determinant. We start with the trick

$$g = e^{\mathrm{tr}\log g^{\mu\nu}}. \tag{4.16}$$

Then, the variation becomes

$$\delta g = e^{\mathrm{tr}\log g^{\mu\nu}}\delta[\mathrm{tr}\log g^{\mu\nu}], \tag{4.17}$$

$$= g\,\mathrm{tr}[g_{\mu\nu}\delta g^{\mu\nu}], \tag{4.18}$$

$$= g g_{\mu\nu}\delta g^{\mu\nu}, \tag{4.19}$$

with the trace tr. Thus, the variation of the root of the negative determinant reads

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}. \tag{4.20}$$

As a result, the minimally coupled part becomes

$$T_{\mu\nu}|_{\xi=0} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\rho\phi\nabla_\rho\phi + \frac{1}{2}g_{\mu\nu}m^2\phi^2. \tag{4.21}$$

The nonminimal coupling is slightly more involved. We start as

$$\delta S_{\mathrm{nonminimal}} = \frac{1}{2}\int_x \xi\phi^2\left[g^{\rho\sigma}\delta R_{\rho\sigma}(x) + \left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu}(x)\right], \tag{4.22}$$

where hereafter we abbreviate $\int_x = \int \mathrm{d}^4x\sqrt{-g}$. Here, we can use the identity (you can find it, for example, in Eq. 4.60 of S. Carroll's notes on relativity)

$$\delta R_{\mu\nu} = \nabla_\rho \delta\Gamma^\rho_{\mu\nu} - \nabla_\nu \delta\Gamma^\lambda{}_{\lambda\mu}. \tag{4.23}$$

Partially integrating, we obtain for the remaining part

$$\frac{1}{2}\int_x \xi\phi^2 g^{\rho\sigma}\delta R_{\rho\sigma}(x) = -\frac{1}{2}\int_x \xi g^{\mu\nu}\left(\nabla_\rho(\phi^2)\delta\Gamma^\rho{}_{\mu\nu} - \nabla_\nu(\phi^2)\delta\Gamma^\lambda{}_{\mu\lambda}\right). \tag{4.24}$$

To obtain the variation of the Christoffel symbols, we use the following shortcut: Construct (Riemann) normal coordinates around a point p. Then, the first derivatives of the metric at that point vanish. Thus, the variation of the Christoffel symbols in that coordinate system reads

$$\delta\Gamma^\rho{}_{\mu\nu}|_p = \frac{1}{2}g^{\rho\sigma}(2\partial_{(\mu}\delta g_{\nu)\sigma} - \partial_\sigma\delta g_{\mu\nu})|_p, \tag{4.25}$$

$$= \frac{1}{2}g^{\rho\sigma}(2\nabla_{(\mu}\delta g_{\nu)\sigma} - \nabla_\sigma\delta g_{\mu\nu})|_p, \tag{4.26}$$

where in the last step we used that in normal coordinates, the Christoffel symbols at p vanish so that $\nabla_\mu|_p = \partial_\mu$. Note now that we could have done this at any point p and that, contrary to $\Gamma^\rho{}_{\mu\nu}$, $\delta\Gamma^\rho{}_{\mu\nu}$ is a tensor (the difference of Christoffel symbols transforms as a tensor) such that Eq. (4.26) is a tensor equation. This implies that Eq. (4.26) holds not only at a point and in normal coordinates, but at all points and every system of coordinates, *i. e.*

$$\delta\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(2\nabla_{(\mu}\delta g_{\nu)\sigma} - \nabla_\sigma\delta g_{\mu\nu}). \tag{4.27}$$

The contracted form of the variation of the Christoffel symbol reads

$$\delta\Gamma^\nu{}_{\mu\nu} = \frac{1}{2}g^{\nu\rho}\nabla_\mu\delta g_{\nu\rho}. \tag{4.28}$$

Thus, we can rewrite the remaining part as

$$\frac{1}{2}\int_x \xi\phi^2 g^{\rho\sigma}\delta R_{\rho\sigma}(x) = \frac{1}{2}\int_x \frac{\xi}{2}g^{\mu\nu}\left(2\delta g_{\rho(\nu}\nabla_{\mu)}\nabla^\rho(\phi^2) - \delta g_{\mu\nu}\nabla_\rho\nabla^\rho(\phi^2) - g^{\nu\rho}\delta g_{\nu\rho}\nabla_\mu\nabla_\nu(\phi^2)\right), \tag{4.29}$$

$$= \frac{1}{2}\int_x \xi\left(\nabla^\mu\nabla^\nu(\phi^2) - g^{\mu\nu}\nabla_\rho\nabla^\rho(\phi^2)\right)\delta g_{\mu\nu} \tag{4.30}$$

Finally, the variation of the metric can be obtained from

$$\delta(\delta^\mu_\nu) = \delta(g^{\mu\rho}g_{\nu\rho}) = \delta g^{\mu\rho}g_{\nu\rho} + g^{\mu\rho}\delta g_{\nu\rho} = 0 \tag{4.31}$$

such that

$$\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}. \tag{4.32}$$

Thus, we can express the variation of the Ricci tensor as

$$\frac{1}{2}\int_x \xi\phi^2 g^{\rho\sigma}\delta R_{\rho\sigma}(x) = \frac{1}{2}\int_x \xi\left(g_{\mu\nu}\nabla_\rho\nabla^\rho(\phi^2) - \nabla_\mu\nabla_\nu(\phi^2)\right)\delta g^{\mu\nu}. \tag{4.33}$$

After all of this tedious algebra, we finally obtain the result

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\rho\phi\nabla^\rho\phi + \frac{m^2}{2}g_{\mu\nu}\phi^2 - \xi\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\phi^2 + \xi\left[g_{\mu\nu}\nabla_\rho\nabla^\rho(\phi^2) - \nabla_\mu\nabla_\nu(\phi^2)\right]. \tag{4.34}$$

Let's go touch some grass.

(**c**) The trace of the stress-energy tensor reads

$$T^{\nu}_{\ \nu} = -\nabla^{\nu}\phi\nabla_{\nu}\phi + 2m^2\phi^2 + \xi R\phi^2 + 6\xi(\phi\Box\phi + \nabla^{\nu}\phi\nabla_{\nu}\phi), \tag{4.35}$$

$$= 6\left(\xi - \frac{1}{6}\right)\nabla^{\nu}\phi\nabla_{\nu}\phi + 2m^2\phi^2 + \xi\phi(6\Box\phi + R\phi). \tag{4.36}$$

But the scalar also satisfies its field equation

$$(\Box + \xi R + m^2)\phi = 0, \tag{4.37}$$

which we can plug in such that

$$T^{\nu}_{\ \nu} = 6\left(\xi - \frac{1}{6}\right)\nabla^{\nu}\phi\nabla_{\nu}\phi + 2m^2\phi^2 + \xi\phi\left[6\left(\frac{1}{6} - \xi\right)R\phi - 6m^2\phi\right], \tag{4.38}$$

$$= 6\left(\xi - \frac{1}{6}\right)\left(\nabla^{\nu}\phi\nabla_{\nu}\phi - \xi R\phi^2\right) + m^2\phi^2\left(2 - 6\xi\right). \tag{4.39}$$

This clearly vanishes when $\xi = 1/6$ and $m = 0$.

Exercise 5: Electromagnetic fields on curved backgrounds

Motivation: Non-conformally coupled scalars are copiously produced in FLRW spacetimes. But how about photons? In other words, is the universe covered in "light" of horizon wavelength?

The Maxwell action on a curved background reads

$$S = -\frac{1}{4}\int \mathrm{d}^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}, \tag{5.1}$$

with the field-strength tensor

$$F_{\mu\nu} = \nabla_{\mu}A_{\nu} - \nabla_{\nu}A_{\mu}, \tag{5.2}$$

and the gauge field A_μ. Instead of going through the whole derivation of particle creation again, we take a shortcut.

(**a**) **Try to do this sub-exercise before reading the remaining ones.**

Work smarter, not harder. Before doing any calculation, think it through: Should Maxwell theory be Weyl invariant in curved spacetime? Why? What does this tell us about photon production in an FLRW background?

Let's now work through the details step by step.

(**b**) The gauge field transforms trivially under Weyl transformations, *i. e.* $A_\mu \to A_\mu$. Calculate how the Maxwell action transforms under Weyl rescalings. Is it invariant?

(**c**) If we rescale the FLRW metric to remove the scale factor, how do Maxwell's equations change? What does this mean for the electromagnetic vacuum in an FLRW background? Is the universe covered in "light" of horizon wave length?

Finally, let's explore how Weyl invariance shows up in the structure of the energy-momentum tensor.

(**d**) Compute the energy-momentum tensor $T^{\mu\nu}$ of the gauge field and show that $T^{\mu}{}_{\mu} = 0$.

(**e**) **Bonus exercise:** What does the tracelessness of the energy-momentum tensor have to do with Weyl invariance? (**Hint:** How does the matter Lagrangian change under a small Weyl rescaling? What would this imply for $T^{\mu}{}_{\mu}$?)

(**a**) Weyl invariance is the invariance under local scale transformations. Thus, if the theory is supposed to be the same at all scales, it cannot introduce absolute scales like dimensionful coupling constants. In other words, the only possible terms which can contribute to a Weyl invariant theory are those with dimensionless couplings. For electromagnetism, there are no minimal-coupling terms with couplings of vanishing dimension which are also gauge invariant (verify that no contractions of one Riemann tensor and one electromagnetic field strength could be viable). Thus, there is no wiggle room for a minimal coupling which may have to be added to render the theory locally Weyl invariant as for the scalar.

This does not imply that the theory is Weyl invariant yet. We can get there keeping in mind that electromagnetism has no scale and is therefore scale invariant in flat space. At the same time, the Maxwell action is made up of two contracted field strength tensors (which are defined with indices down). Their contraction requires two inverse metric tensors, whose behaviour under scale transformations balances exactly the one from the metric determinant. In other words, in flat space, the gauge field transforms trivially under scale transformations.

Recall that in exercise 4 the derivatives in the kinetic term of the scalar spoiled local Weyl invariance by introducing derivatives of the conformal factor. This cannot happen for the electromagnetic field (whose kinetic term is exactly the Maxwell action) because the field strength transforms trivially. Thus, the Maxwell action has to be locally Weyl invariant.

If we take seriously what we learned in the lecture, this indicates that the electromagnetic field does not "see" the cosmological evolution. This would indicate that there can be no photon production in cosmology.

(**b**) We know that $A_\mu \to A_\mu$. Besides, the covariant derivatives in field strength receive no gravitational contributions because

$$F_{\mu\nu} = 2\nabla_{[\mu}A_{\nu]} = 2\partial_{[\mu}A_{\nu]} - \Gamma^{\rho}{}_{[\mu\nu]} = 2\partial_{[\mu}A_{\nu]}, \tag{5.3}$$

where we used that the Levi-Civita connection is torsionless (aka that $\Gamma^{\rho}{}_{\mu\nu}$ is symmetric in (μ,ν)). As a result, we find that $F_{\mu\nu} \to F_{\mu\nu}$. At the same time $g^{\mu\nu} \to \Omega^{-2} g^{\mu\nu}$ and $g \to \Omega^8 g$. So the action

$$S = -\frac{1}{4}\int \mathrm{d}^4x\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} = -\frac{1}{4}\int \mathrm{d}^4x\sqrt{-\bar{g}}\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{5.4}$$

is indeed invariant.

(**c**) Maxwell's equations in a curved background read

$$\nabla^{\nu}F_{\mu\nu} = 0. \tag{5.5}$$

For the covariant derivative, we have

$$\nabla^{\nu}F_{\mu\nu} = \partial^{\nu}F_{\mu\nu} - g^{\nu\sigma}\Gamma^{\rho}{}_{\mu\sigma}F_{\rho\nu} - g^{\nu\sigma}\Gamma^{\rho}{}_{\sigma\nu}F_{\mu\rho}. \tag{5.6}$$

Recall that the Christoffel symbols transform under Weyl transformations as given in Eq. (4.5), implying

$$\bar{\Gamma}^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\mu\nu} = \frac{2\delta^{\rho}_{(\mu}\nabla_{\nu)}\Omega - g_{\mu\nu}\nabla^{\rho}\Omega}{\Omega}. \tag{5.7}$$

$$\begin{aligned}\delta\left(\nabla^{\nu}F_{\mu\nu}\right) =&\bar{\nabla}^{\nu}\bar{F}_{\mu\nu}-\nabla^{\nu}F_{\mu\nu}, && (5.8)\\ =&-\frac{2\nabla_{(\sigma}\Omega F_{\mu)}{}^{\sigma}-F_{\rho\mu}\nabla^{\rho}\Omega}{\Omega}+\frac{2F_{\mu\rho}\nabla^{\rho}\Omega}{\Omega}, && (5.9)\\ =&\frac{-F_{\mu\sigma}\nabla^{\sigma}\Omega+F_{\rho\mu}\nabla^{\rho}\Omega+2F_{\mu\rho}\nabla^{\rho}\Omega}{\Omega}, && (5.10)\\ =&0, && (5.11)\end{aligned}$$

where we used the antisymmetry of the field strength tensor. Thus, as expected, they remain unchanged.

Thus, the mode equation for photons is equivalent to the one in flat space. Then, there is a unique vacuum. The universe is therefore not filled with light.

(**d**) The energy-momentum tensor is defined in Eq. (4.15). We have to vary the Maxwell action with respect to the inverse metric. We thus have

$$\begin{aligned}\delta S=&-\frac{1}{4}\int \mathrm{d}^4x\delta\left(\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}\right), && (5.12)\\ =&-\frac{1}{4}\int \mathrm{d}^4x\sqrt{-g}\left[\left(2F_{\mu\rho}F^{\rho}_{\ \nu}-\frac{1}{2}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)\delta g^{\mu\nu}\right]. && (5.13)\end{aligned}$$

There is no contribution from the variation of the field strength because due to Eq. (5.3) it is independent of the metric. Then, the energy-momentum tensor reads

$$T_{\mu\nu}=F_{\mu\rho}F^{\rho}_{\ \nu}-\frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}. \tag{5.14}$$

Contracting the two indices, we obtain the trace

$$T^{\mu}_{\ \mu}=0. \tag{5.15}$$

(**e**) Consider some matter action $\mathcal{L}_{\text{matter}}(g_{\mu\nu},\Phi^a)$, with some set of matter fields Φ_a, where the index a enumerates the fields. These can be vectors, spinors, scalars, apples ... If we apply an infinitesimal Weyl transformation such that $\Omega^2=1+\delta\omega$, the action changes as

$$\delta S=\int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{2}T_{\mu\nu}\delta g^{\mu\nu}+\frac{\delta\mathcal{L}_{\text{matter}}}{\delta\Phi^a}\delta\Phi^a+\text{total derivatives}\right). \tag{5.16}$$

Applying the equations of motion, removing total derivatives and plugging in $\delta g^{\mu\nu}=-g^{\mu\nu}\delta\omega$, we obtain the transformation

$$\delta S=\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}T^{\mu}_{\ \mu}\delta\omega. \tag{5.17}$$

Hence, the change in the action vanishes iff $T^{\mu}_{\ \mu}=0$. In plain English, a theory is Weyl invariant if the energy-momentum tensor has vanishing trace.

Exercise 6: Impact of general nonminimal coupling on particle production

Motivation: Last week, we found that minimally coupled scalars are generically produced in FLRW backgrounds. This does not happen for conformally coupled scalars. Here we estimate what happens for general nonminimal coupling.

Consider a nonminimally coupled massive scalar whose action is given by Eq. (4.1).

(a) Compute the equation of motion for the scalar in an FLRW background, and redefine the field $\phi \to \chi = a(\eta)\phi$ such that the friction term ($\sim \phi'$) disappears. You should obtain that the nonminimal coupling gives you an additional contribution to the effective mass.

Sanity check: What happens in the limit $\xi \to 1/6$, $m \to 0$?

(b) Assume that the background is changing slowly and consider modes with small wavelength. Find out when these two assumptions are actually equivalent.

(c) Start in the adiabatic vacuum at some conformal time $\eta = \eta_0$, namely $|0_{\mathrm{ad},\eta_0}\rangle$, and look at the resulting state at a time $\eta = \eta_0 + \Delta\eta$. Try to get at some qualitative properties of the average particle-number density $\langle 0_{\mathrm{ad},\eta_0}|n_k|0_{\mathrm{ad},\eta_0}\rangle(\eta_0 + \Delta\eta)$ without calculating it. Sketch how you expect the particle-number density to depend on ξ. (**Hint:** Keep in mind that particles are not produced if the background is static, and that in terms of some complex time-dependent Bogolyubov parameter $\beta_k(\eta)$

$$\langle 0_{\mathrm{ad},\eta_0}|n_k|0_{\mathrm{ad},\eta_0}\rangle(\eta) = |\beta_k(\eta)|^2. \tag{6.1}$$

You can find inspiration in exercise 1.)

The scalar satisfies the equation of motion

$$(\Box + m^2 + \xi R)\phi = 0. \tag{6.2}$$

Specifying to a flat FLRW spacetime in the conformal slicing (with conformal time η as time) and a field

$$\phi = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \phi_{\mathbf{k}}(\eta) e^{-i\mathbf{k}\mathbf{x}}, \tag{6.3}$$

the equation of motion takes the form

$$\phi''_{\mathbf{k}} + 2\frac{a'}{a}\phi' + \left(k^2 + m^2 a^2 + \xi R\right)\phi_{\mathbf{k}} = 0. \tag{6.4}$$

Defining a new field $\chi \equiv \phi a$, and correspondingly $\chi_{\mathbf{k}} = \phi_{\mathbf{k}} a$, we obtain field equation

$$\chi''_{\mathbf{k}} + \left[k^2 + a^2 m^2 + 6a^2\left(\xi - \frac{1}{6}\right)R\right]\chi_{\mathbf{k}} \equiv \chi''_{\mathbf{k}} + \omega_k^2(\eta)\chi_{\mathbf{k}} = 0, \tag{6.5}$$

where we used that in spatially flat FLRW $R = 6a''/a^3$. Therefore, we find a modification to the effective mass, which, including nonminimal coupling, now reads

$$m_{\text{eff}}^2 = a^2\left[m^2 + 6\left(\xi - \frac{1}{6}\right)R\right]. \tag{6.6}$$

When $\xi = 1/6$, $m = 0$, the effective mass vanishes and the field equation for χ becomes that of a massless scalar in Minkowski spacetime.

(b) If the background is changing slowly, we have (see exercise 3)

$$\frac{\omega_k'}{\omega_k^2} \ll 1. \tag{6.7}$$

For modes with small wavelength $k^2 \gg m_{\text{eff}}^2$ during the whole evolution considered such that

$$\omega_k \simeq k + \frac{m_{\text{eff}}^2}{2k}. \tag{6.8}$$

Thus, for its time derivative, we obtain

$$\frac{\omega_k'}{\omega_k^2} \simeq \frac{(m_{\text{eff}}^2)'}{2k^3}. \tag{6.9}$$

Thus, for the adiabatic approximation not to apply to small-wavelength modes, the scale factor has to satisfy

$$a^2(m^2 + (6\xi - 1)R) \ll k^2, \qquad 2\mathcal{H}a^2(m^2 + (6\xi - 1)R) + a^2(6\xi - 1)R' \geq k^3 \tag{6.10}$$

over the whole evolution (here $\mathcal{H} = a'/a$). Thus, (neglecting unexpected cancellations) either $\mathcal{H} \gg k$ or $a^2R' \sim \mathcal{H}'' + 2\mathcal{H}'\mathcal{H} \geq k^3/a^2$, while at the same time $a^2R \sim \mathcal{H}' + \mathcal{H}^2 \ll k^2$. In other words, some derivatives of the scale factor have to be very large, while others have to be very small. Unless this is the case (which is rarely so, especially when evolving over longer times), large modes experience a slowly varying background.

(**c**) Let's start with the hint: The expected number density will be the squared norm of some complex Bogolyubov coefficient β_k. We have to estimate that coefficient. There is no particle creation if there is no time evolution. Thus, β_k at first order has to be a function of ω_k'/ω_k^2, the first kind of correction in the adiabatic approximation. Besides, it has to have an oscillating component which stems from the mixing of positive- and negative-frequency modes. According to the WKB-approximation, oscillating phases generically have arguments proportional to $\int \omega_k(\eta')\mathrm{d}\eta'$. Inspired by Eq. (1.4), we take the oscillating function to be a sinus. Besides, the particle density is the integrated number of created particles (they don't just disappear from one moment to the other) – by dimensional analysis the integral has to be balanced by an additional power of ω_k. Thus, we arrive at the estimate for the Bogolyubov coefficient

$$\beta_k = a \int_{\eta_0}^{\eta_0+\Delta\eta} \mathrm{d}\bar{\eta} \frac{\omega_k'(\bar{\eta})}{\omega_k(\bar{\eta})} \sin\left[b \int_{\eta_0}^{\bar{\eta}} \mathrm{d}\tilde{\eta}\omega_k(\tilde{\eta})\right], \tag{6.11}$$

for a complex coefficient a and a real coefficient b, both of which are expected to be of order 1 in absolute value. Note how close this estimate is to Eq. (1.4) even though the evolution there is not adiabatic (the box-like behaviour produces delta-functions in ω_k' which definitely break the adiabatic approximation). At large k, we obtain

$$\beta_k \simeq a \int_{\eta_0}^{\eta_0+\Delta\eta} \mathrm{d}\bar{\eta} \frac{m_{\text{eff}}^{2\prime}(\bar{\eta})}{k^2} \sin\left[b \int_{\eta_0}^{\bar{\eta}} \mathrm{d}\tilde{\eta}\omega_k(\tilde{\eta})\right]. \tag{6.12}$$

Say $\Delta\eta$ is small enough such that the adiabatic approximation holds integrated over the whole evolution (see Eq. (3.30)). Then, we can take the effective mass out of the integral and obtain

$$\beta_k \simeq a \frac{m_{\text{eff}}^{2\prime}(\eta_0)}{k^2} \int_{\eta_0}^{\eta_0+\Delta\eta} \mathrm{d}\bar{\eta} \sin\left[b(\bar{\eta} - \eta_0)\omega_k(\tilde{\eta})\right], \tag{6.13}$$

$$\simeq a \frac{m_{\text{eff}}^{2\prime}(\eta_0)}{k^3} \left[1 - \cos(b\Delta\eta\omega_k(\eta_0))\right]. \tag{6.14}$$

The number density then reads

$$\langle 0_{\text{ad},\eta_0}|n_{\mathbf{k}}|0_{\text{ad},\eta_0}\rangle = 4|a|^2 \frac{|m_{\text{eff}}^{2\prime}|^2}{k^4} \sin^4(b\Delta\eta\omega_k(\eta_0)/2). \tag{6.15}$$

Note that, here, the oscillating function was a choice. Depending on the specific situation, it can be more complicated. However, the qualitative behaviour (oscillating, proportional to $|m_{\text{eff}}^{2\prime}|^2/k^4$) is rather robust, unless $a = 0$ which would imply that the corrections are of higher order. In particular, at a fixed moment in time, the scaling with the nonminimal-coupling parameter is like

$$\langle 0_{\text{ad},\eta_0}|n_{\mathbf{k}}|0_{\text{ad},\eta_0}\rangle \sim |6\xi - 1 + \text{const linear in } m^2|^2. \tag{6.16}$$

Thus, the number density scales quadratically with ξ. Of course, if $m = 0$, the particle-number density vanishes for $\xi = 1/6$. As an example we plot the scaling $|6\xi - 1 + 6m^2|$ in fig. 1.

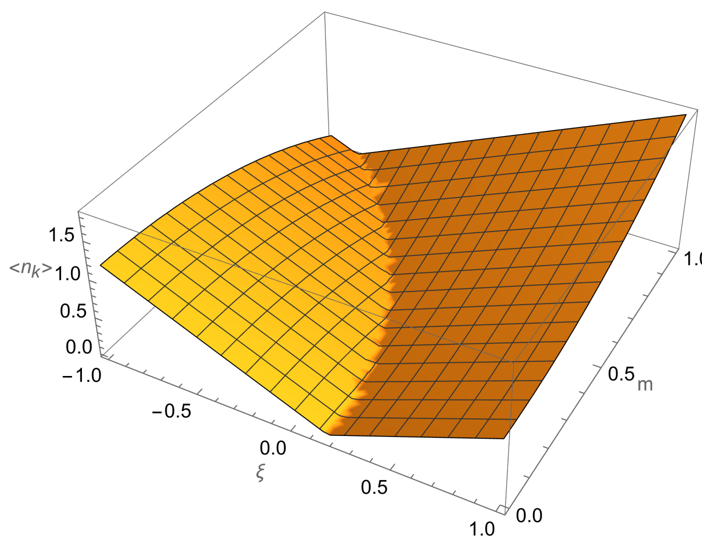

Figure 1: Exemplary scaling of number density with nonminimal-coupling parameter and mass. The exact value of $\langle n_{\mathbf{k}} \rangle$ as it is plotted here has no physical meaning because we did not derive an exact result.