Quantum field theory in curved spacetime

Assignment 3 – May 12

Exercise 7: Uniformly accelerated observers aka Rindler space

Motivation: Before we explore how the Minkowski vacuum appears to uniformly accelerated observers, we first need to understand the dynamics of accelerated motion in special relativity. As we'll find out, this brings up some of the exciting concepts usually reserved for general relativity.

First things first: For this and the following exercise, we do not need any general relativity. In the end, we'll just use a weird coordinate system to parametrize Minkowski spacetime. Don't believe the detractors who say special relativity can't describe non-inertial motion!
For simplicity, let's start in two-dimensional Minkowski space described in terms of Cartesian coordinates such that

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{7.1}$$

We want to describe a timelike observer moving along a uniformly accelerated trajectory. Parameterize the curve by the proper time of the observer τ. We denote their four-velocity, *i. e.* the tangent vector along the curve, as u^μ with norm $u^\mu u_\mu = -1$. We can, thus, define the proper acceleration as

$$a^\mu = u^\nu \nabla_\nu u^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu{}_{\nu\rho} u^\nu u^\rho, \tag{7.2}$$

which we readily identify as the left-hand side of the geodesic equation. As expected, if $a^\mu \neq 0$, motion is not geodesic/inertial.

(a) Show that the proper acceleration is spacelike. (**Hint:** Work in Cartesian coordinates)

(b) Assume that the acceleration is uniform. Then, the norm of the proper acceleration is constant. Denote it as $a^\mu a_\mu \equiv \mathfrak{a}^2 = \text{const}$. Construct the unique four-velocity that yields the correct norm for both the velocity and the acceleration. (**Hint:** Relative to an observer at rest, constant acceleration is like a boost with time-dependent boost parameter.)

(c) Find a parametrization of the curve $(\gamma(\tau) = (t(\tau), x(\tau))$, and show that uniformly accelerated observers move on hyperbolae in spacetime, *i. e.* $x^2 - t^2 = \mathfrak{a}^{-2}$. (**Hint:** For the second part, try to eliminate any explicit dependence on τ.)

(d) Find a coordinate system (η, ρ) such that the parabolic motion is realized by setting $\rho = \mathfrak{a}^{-1} = \text{const}$. You should obtain

$$\mathrm{d}s^2 = -\rho^2 \mathrm{d}\eta^2 + \mathrm{d}\rho^2. \tag{7.3}$$

Show that the new coordinate system amounts to the rest frame of the accelerated particle. (**Hint:** Think of an analogue of polar coordinates.)

Sanity check: Make sure that $u^\mu u_\mu = -1$ continues to be the case in the new coordinate system.

(**e**) For QFT (see next exercise), it is useful to coordinatize the space using the Rindler proper time $\tau = a^{-1}\eta$, and the position coordinate $\xi = \mathfrak{a}^{-1}\log \mathfrak{a}\rho$. Show that the resulting metric reads

$$\mathrm{d}s^2 = e^{2\mathfrak{a}\xi}(-\mathrm{d}\tau^2 + \mathrm{d}\xi^2). \tag{7.4}$$

Why could the metric be particularly interesting in this shape?

(**f**) Draw the Rindler trajectories in a Cartesian coordinate system for different values of $\mathfrak{a}$. What happens in the limit $\mathfrak{a} \to \infty$? Draw the corresponding limiting surface in your diagram as well as light-cones (light rays=45°-lines) emanating from points for which $t > x$. Can light from these points reach accelerated observers? How do we interpret this result?

Bonus question: Do the Rindler coordinates cover the whole Minkowski space?

(**g**) How do we generalize to four dimensions? A short argument should be sufficient.

(**a**) In Cartesian coordinates, the Christoffel symbols vanish. Thus, we have

$$a^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}. \tag{7.5}$$

If we take the derivative with respect to τ of the norm of the four-velocity, we obtain

$$\frac{\mathrm{d}(u^\mu u_\mu)}{\mathrm{d}\tau} = 2u_\mu \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = 2u^\mu a_\mu = 0. \tag{7.6}$$

Thus, the proper acceleration is normal to the four-velocity. Since the four-velocity is timelike, the proper acceleration has to be spacelike.

(**b**) A boost with rapidity η applied to a rest-frame four-velocity ($u^\mu = (1, 0)$) reads

$$u^\mu = (\cosh \eta, \sinh \eta). \tag{7.7}$$

One can check that indeed $u^\mu u_\mu = -\cosh^2 \eta + \sinh^2 \eta = -1$. Let's make $\eta = \eta(\tau)$ time dependent to indicate the acceleration. Then, the proper acceleration reads

$$a^\mu = \dot{\eta}(\sinh \eta, \cosh \eta). \tag{7.8}$$

Its norm reads

$$a^\mu a_\mu = \dot{\eta}^2(\cosh^2 \eta - \sinh^2 \eta) = \dot{\eta}^2 = \mathfrak{a}^2. \tag{7.9}$$

In other words, up to an irrelevant integration constant $\eta = \mathfrak{a}\tau$. Thus, the four-velocity reads

$$u^\mu = (\cosh \mathfrak{a}\tau, \sinh \mathfrak{a}\tau). \tag{7.10}$$

(**c**) Integration of the four-velocity yields $\gamma = (t(\tau), x(\tau)) = \mathfrak{a}^{-1}(\sinh \mathfrak{a}\tau, \cosh \mathfrak{a}\tau)$. We can eliminate any dependence on τ by considering the combination

$$x(\tau)^2 - t(\tau)^2 = \mathfrak{a}^{-2}. \tag{7.11}$$

This confirms that the motion traces a hyperbola. This is the Minkowski-space analogue of a circle: a curve of constant proper distance from the origin.

(**d**) By analogy with spherical coordinates, we define the hyperbolic coordinates $x'^\mu = (\eta, \rho)$, also called Rindler coordinates,[1]

$$t = \rho \sinh \eta, \qquad x = \rho \cosh \eta. \tag{7.12}$$

[1] The variable name η is chosen on purpose by a slight abuse of notation.

Then, uniformly accelerated motion simply amounts to $\rho(\tau) = \mathfrak{a}^{-1} = \text{const}$. We will derive the parametrization of $\eta(t)$ below.

The Jacobian of the transformation from Cartesian coordinates to Rindler coordinates reads

$$\frac{\partial x^\mu}{\partial x'^\nu} = \begin{pmatrix} \rho\cosh\eta & \rho\sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix}, \qquad \frac{\partial x'^\mu}{\partial x^\nu} = \begin{pmatrix} \rho^{-1}\cosh\eta & -\sinh\eta \\ -\rho^{-1}\sinh\eta & \cosh\eta \end{pmatrix}. \tag{7.13}$$

Then, the metric reads

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{7.14}$$
$$= g_{\mu\nu}\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma}\mathrm{d}x'^\rho\mathrm{d}x'^\sigma \tag{7.15}$$
$$= g'_{\mu\nu}\mathrm{d}x'^\mu\mathrm{d}x'^\nu \tag{7.16}$$
$$= -\rho^2\mathrm{d}\eta^2 + \mathrm{d}\rho^2. \tag{7.17}$$

In Rindler coordinates, the four-velocity of the uniformly accelerated observer reads

$$u = u^\mu\partial_\mu, \tag{7.18}$$
$$= u^\mu\frac{\partial x'^\nu}{\partial x^\mu}\partial'_\nu|_{\rho=\mathfrak{a}^{-1},\eta=\eta(\tau)}, \tag{7.19}$$
$$= \rho^{-1}\left(\cosh\mathfrak{a}\tau\cosh\eta - \sinh\mathfrak{a}\tau\sinh\eta\right)\partial_\eta$$
$$+ \left(\cosh\mathfrak{a}\tau\sinh\eta - \sinh\mathfrak{a}\tau\cosh\eta\right)\partial_\rho|_{\rho=\mathfrak{a}^{-1},\eta=\eta(\tau)}. \tag{7.20}$$

We recover the expected form of the velocity if $\eta(\tau) = \mathfrak{a}\tau$ such that

$$u = \mathfrak{a}\partial_\eta. \tag{7.21}$$

Indeed, the velocity has no component along the spatial direction. In other words, this coordinate system tracks the proper time of (this could have also been seen from $\eta = \mathfrak{a}\tau$) and defines the rest frame comoving with the uniformly accelerated observer.

The norm of the four-velocity then becomes

$$u^\mu u^\nu g'_{\mu\nu}|_{\rho=\mathfrak{a}^{-1},\eta=\mathfrak{a}\tau} = -\mathfrak{a}^2\rho^2|_{\rho=\mathfrak{a}^{-1}} = -1. \tag{7.22}$$

(**e**) The differentials after the coordinate transformation read

$$\mathrm{d}\eta = \mathfrak{a}\mathrm{d}\tau, \qquad \mathrm{d}\rho = e^{\mathfrak{a}\xi}\mathrm{d}\xi. \tag{7.23}$$

Thus, the metric clearly reads

$$\mathrm{d}s^2 = e^{2\mathfrak{a}\xi}(\mathrm{d}\tau^2 - \mathrm{d}xi^2). \tag{7.24}$$

In this form, the metric is conformally flat. This is useful especially when considering Weyl invariant theories because those satisfy Cartesian-like equations of motion also for Rindler coordinates.

(**f**) I plot the trajectories of different Rindler observers in fig. 2. In the limit $\mathfrak{a} \to \infty$, the trajectories asymptotically approach the light cone emanating from the origin which satisfies $x = t$. As light cones do not cross in Minkowski spacetime, no light can reach the accelerated observers from points where $t > x$. Such a surface is called a horizon. This resembles the event horizon of a black hole, just that (in more than two dimensions) it's infinitely large. Thus whatever happens to accelerating observers is analogous to what happens to observers close to a black hole.

Having a closer look at the metric in Rindler coordinates (Eq. (7.17)), its determinant vanishes at $\rho = 0$, *i. e.* at the horizon. This implies that there is a singularity there. As there is no singularity in Minkowski spacetime in Cartesian coordinates, this has to be a coordinate singularity. Indeed, this is

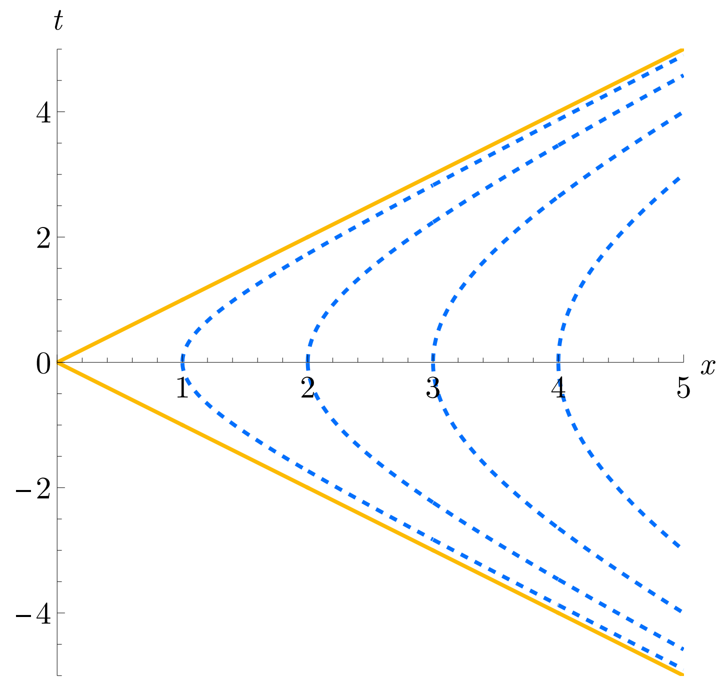

Figure 2: Trajectories of Rindler observers in Cartesian coordinates in two-dimensional Minkowski spacetime. The trajectories for $\mathfrak{a} = 1/n$ for $n = 1, 2, 3, 4$ are represented by blue dashed lines, while the yellow line is approached in the limit $\mathfrak{a} \to \infty$.

the same coordinate singularity you have in the origin in polar coordinates. It also indicates that you cannot describe Minkowski spacetime beyond the horizon in terms of Rindler coordinates as we defined them. While it is possible to extend the coordinates to negative ρ such that the mirror image of the region covered in fig. 2 can be described, the whole light-cone emanating from the origin requires a modified coordinate system. In the rest of the sheet we will concentrate on the so-called Rindler patch, *i. e.* the part $\rho > 0$.

(**g**) The uniformly accelerated observer is uniformly accelerated along one spatial direction. Without loss of generality, we can assume this axis to be the z-axis (with Cartesian coordinates (t, x, y, z)) such that the metric assumes the form

$$\mathrm{d}s^2 = -\rho^2 \mathrm{d}\eta^2 + \mathrm{d}\rho^2 + \mathrm{d}x^2 + \mathrm{d}y^2. \tag{7.25}$$

Exercise 8: Unruh effect

Motivation: Particle creation is not just an effect of curved spacetime. Accelerated observers in flat spacetime are embedded into a thermal bath of particle-antiparticle pairs. Let's find out how.

We want to quantize a massless Klein-Gordon field in two-dimensional Minkowski spacetime sliced by the set of Rindler coordinates (τ, ξ), see Eq. (7.4) for the metric. The action for the massless scalar reads

$$S = -\frac{1}{2} \int \mathrm{d}x \sqrt{-g} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi. \tag{8.1}$$

The corresponding equation of motion for the scalar reads

$$\Box \phi = g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = 0. \tag{8.2}$$

We will quantize the solutions to this equation. This task is analogous to quantization in two-dimensional FLRW spacetime – in FLRW you have a time-dependent conformal factor in the metric. Now you have a position-dependent conformal factor in the metric.

(a) Express the equation of motion in Rindler coordinates (τ, ξ). The result should look analogous to the equation of motion in Cartesian coordinates. Why is it so simple?

(b) Solve the equation of motion.

Next we need to define what particles and antiparticles are. By convention, we write down the mode expansion in terms of orthonormal solutions as

$$\phi = \int_0^\infty \mathrm{d}\omega \left(a_\omega v_\omega + a_\omega^\dagger v_\omega^* \right), \tag{8.3}$$

where the creation and annihilation operators satisfy the usual commutation relations, and v_ω is a set of orthonormal solutions of the equations of motion. We generically choose $\omega > 0$ because throughout this exercise, we only consider right-moving solutions. This is simpler, and including left-movers wouldn't change the result.

For those who haven't done GR yet, the following may be a lot. If all this lingo makes no sense to you, just skip to sub-exercise (d). On general curved backgrounds (or backgrounds described by weird curvilinear coordinates), the space of solutions to the Klein-Gordon equation possesses an inner product on hypersurfaces of constant time Σ (whatever crazy time one may choose to work with). Given unitary time evolution along the normal vector to the surfaces n^μ (which is, of course, timelike and normalized), this inner product is conserved, thus allowing to find orthonormal solutions for all times. The Klein-Gordon inner product for $d+1$-dimensional backgrounds reads

$$(v_1, v_2) = -i \int_\Sigma \mathrm{d}^d x \sqrt{h} \left(v_1^* n^\mu \nabla_\mu v_2 - v_2^* n^\mu \nabla_\mu v_1 \right), \tag{8.4}$$

where the integral is over a surface of constant time, and h denotes the determinant of the induced metric.

(c) **Try to do this exercise before reading on:** What is the dimensionality of the hypersurface and what are h and n^μ for $1+1$-dimensional Minkowski spacetime sliced by Rindler time? Construct the Klein-Gordon inner product.

Orthonormal solutions v_ω satisfy the conditions

$$(v_\omega, v_{\omega'}) = \delta(\omega - \omega'), \qquad (v_\omega, v_{\omega'}^*) = 0. \tag{8.5}$$

(d) The Klein-Gordon inner product you should have obtained in sub-exercise **(c)** reads

$$(v_1, v_2) = -i \int_{-\infty}^{\infty} \mathrm{d}\xi (v_1^* \partial_\tau v_2 - v_2 \partial_\tau v_1^*). \tag{8.6}$$

Show that the (by convention) right-moving negative-frequency solutions

$$v_\omega = \frac{e^{i\omega(\tau - \xi)}}{\sqrt{4\pi\omega}} \tag{8.7}$$

are orthonormal (recall that $\omega > 0$). Thus, we can express the quantized field in terms of its mode expansion

$$\phi = \int_0^\infty \frac{\mathrm{d}\omega}{\sqrt{4\pi\omega}} \left[a_\omega e^{i\omega(\tau-\xi)} + a_\omega^\dagger e^{-i\omega(\tau-\xi)} \right]. \tag{8.8}$$

The Klein-Gordon inner product helps us with more than just constructing the mode expansion. It can also give us the overlap between modes of different mode expansions.
If you remember our last exercise sheet, distinct bases of mode functions are related by Bogolyubov coefficients. The right-moving, negative-frequency Cartesian mode functions read

$$u_k = \frac{e^{ik(t-x)}}{\sqrt{4\pi k}}, \tag{8.9}$$

for some $k > 0$ (again we only consider right-moving modes) such that the Cartesian mode expansion reads

$$\phi = \int_0^\infty \mathrm{d}k \left(b_k u_k + b_k^\dagger u_k^* \right), \tag{8.10}$$

with creation and annihilation operators b_k, $b_k^\dagger$. The inertial vacuum (*i. e.* the vacuum from the Cartesian mode expansion) $|0_{\text{inert}}\rangle$ is defined such that it is annihilated by b_k.
The inertial mode functions can be expressed in terms of the Rindler mode functions as

$$u_k = \int_0^\infty \mathrm{d}\omega \left(\alpha_{k\omega} v_\omega + \beta_{k\omega} v_\omega^* \right), \tag{8.11}$$

for some Bogolyubov coefficients $\alpha_{k\omega}$, $\beta_{k\omega}$. Make sure you understand why this works. Thus, generically they mix positive- and negative-frequency Rindler modes: The inertial vacuum is not necessarily empty for non-inertial observers. As we learned last week, we can express the one vacuum in terms of states of a different basis by computing the Bogolyubov coefficients. Let's see how this works exactly.

(e) Using the orthogonality relations in Eq. (8.5), show that

$$\alpha_{k\omega} = (v_\omega, u_k), \qquad \beta_{k\omega} = (v_\omega^*, u_k). \tag{8.12}$$

(f) Last week, we found in an analogous problem that the vacuum state in one basis may be populated with particles in a different one, and that the particle-number density depends on the Bogolyubov coefficients. Therefore, show that

$$\alpha_{k\omega} = \frac{\omega}{2\pi\sqrt{\omega k \mathfrak{a}}} \mathfrak{a}^{\frac{i\omega}{\mathfrak{a}}} k^{-\frac{i\omega}{\mathfrak{a}}} e^{\frac{\pi\omega}{2\mathfrak{a}}} \Gamma\left(\frac{i\omega}{\mathfrak{a}}\right), \tag{8.13}$$

$$\beta_{k\omega} = -\frac{\omega}{2\pi\sqrt{\omega k \mathfrak{a}}} \mathfrak{a}^{-\frac{i\omega}{\mathfrak{a}}} k^{\frac{i\omega}{\mathfrak{a}}} e^{-\frac{\pi\omega}{2\mathfrak{a}}} \Gamma\left(-\frac{i\omega}{\mathfrak{a}}\right). \tag{8.14}$$

(**Hint:** Keep in mind that ϕ is a scalar field and transforms as such. You may use that the Laplace transform of a power is related to the Gamma function as

$$\int_0^\infty z^{s-1} e^{-\lambda z} \mathrm{d}z = \lambda^{-s}\Gamma(s), \tag{8.15}$$

and analytically continue to complex s and λ.)

Work smarter, not harder. Do you have to compute both $\alpha_{k\omega}$ and $\beta_{k\omega}$ individually or are they somehow related?

(**g**) Show that the number density in the inertial vacuum can be expressed as (**Note that this expression was ordered wrongly in the original sheet. This mistake was propagating through the exercise.**)

$$n_\omega = V^{-1}\langle 0_{\text{inert}}|a^\dagger_\omega a_\omega|0_{\text{inert}}\rangle = \int_0^\infty \mathrm{d}k|\beta_{k\omega}|^2, \tag{8.16}$$

with the volume of the spacelike slice V, which as seen in earlier exercises, is required for regularization.

(**h**) The integral in Eq. (8.16) is divergent and would have to be regularized. Instead of evaluating the integral explicitly, we use a trick. First show that

$$n_\omega = -1 + \int_0^\infty \mathrm{d}k|\alpha_{k\omega}|^2. \tag{8.17}$$

Next, having a closer look at Eqs. (8.13) and (8.14), express $\beta_{k\omega}$ in terms of $\alpha_{k\omega}$. This should allow you to compute n_ω without integrating to obtain

$$n_\omega = \frac{1}{e^{\frac{2\pi\omega}{\mathfrak{a}}} - 1}. \tag{8.18}$$

Interpret this result: What type of distribution is this?

(**a**) In Rindler coordinates, the equation of motion reads

$$(\partial_\tau^2 - \partial_\xi^2)\phi = 0. \tag{8.19}$$

This equation is so simple just because the massless scalar is Weyl invariant in two dimensions. Thus, we can transform away the conformal factor without implications for the scalar. This is fully analogous to the conformally coupled scalar and the Maxwell field in four dimensions on the last sheet.

(**b**) Solutions to the equation of motion Eq. (8.19) are equivalent to the Cartesian solution. Thus, the most general solution reads

$$\phi = Ae^{i\omega(\tau+\xi)} + Be^{i\omega(\tau-\xi)} + Ce^{-i\omega(\tau+\xi)} + De^{-i\omega(\tau-\xi)}, \tag{8.20}$$

for some complex constants A, B, C and D. That ϕ is a real scalar imposes the constraints

$$C = A^*, \qquad D = B^*, \tag{8.21}$$

such that we obtain the general solution

$$\phi = Ae^{i\omega(\tau+\xi)} + Be^{i\omega(\tau-\xi)} + \text{h. c.} \tag{8.22}$$

Note here that the direction of motion is encoded in the sign of ξ such that generically $\omega \geq 0$.

(**c**) The hypersurface is one-dimensional, *i. e.* a line. As shown in exercise 7 (**d**), the normal equals the Rindler velocity $n = u = \rho^{-1}\partial_\eta = e^{-\mathfrak{a}\xi}\partial_\tau$, and the induced "metric" on the line is $\mathrm{d}s^2_{(1)} = e^{2\mathfrak{a}\xi}\mathrm{d}\xi^2$, so $h = e^{2\mathfrak{a}\xi}$. Thus, the factors from the determinant of the induced metric and the normal vector cancel, and the Klein-Gordon inner product reads

$$(\phi_1, \phi_2) = -i\int_{-\infty}^{\infty} \mathrm{d}\xi(\phi_1^*\partial_\tau\phi_2 - \phi_2\partial_\tau\phi_1^*). \tag{8.23}$$

(**d**) For the negative-frequency solutions Eq. (8.7), we obtain the inner products

$$(v_\omega, v_{\omega'}) = \frac{\omega+\omega'}{4\pi\sqrt{\omega\omega'}} e^{-i(\omega-\omega')\tau} \int_{-\infty}^{\infty} \mathrm{d}\xi e^{i(\omega-\omega')\xi}, \tag{8.24}$$

$$= \frac{\omega'+\omega}{2\sqrt{\omega\omega'}} e^{-i(\omega-\omega')\tau} \delta(\omega-\omega'), \tag{8.25}$$

$$= \delta(\omega-\omega'), \tag{8.26}$$

$$(v_\omega, v^*_{\omega'}) = \frac{\omega+\omega'}{4\pi\sqrt{\omega\omega'}} e^{-i(\omega+\omega')\tau} \int_{-\infty}^{\infty} \mathrm{d}\xi e^{i(\omega+\omega')\xi}, \tag{8.27}$$

$$= \frac{\omega'+\omega}{2\sqrt{\omega\omega'}} e^{-i(\omega+\omega')\tau} \delta(\omega+\omega'), \tag{8.28}$$

$$= 0 \tag{8.29}$$

as long as $\omega, \omega' > 0$.

(**e**) We start on the right-hand side of Eq. (8.12) and show that it equals the left-hand side:

$$(v_\omega, u_k) = \int_0^\infty \mathrm{d}\omega' \left[\alpha_{k\omega'}(v_\omega, v_{\omega'}) + \beta_{k\omega'}(v_\omega, v^*_{\omega'})\right], \tag{8.30}$$

$$= \int_0^\infty \mathrm{d}\omega' \alpha_{k\omega'} \delta(\omega-\omega'), \tag{8.31}$$

$$= \alpha_{k\omega}. \tag{8.32}$$

Analogously, we obtain

$$(v^*_\omega, u_k) = \int_0^\infty \mathrm{d}\omega' \left[\alpha_{k\omega'}(v^*_\omega, v_{\omega'}) + \beta_{k\omega'}(v^*_\omega, v^*_{\omega'})\right], \tag{8.33}$$

$$= \int_0^\infty \mathrm{d}\omega' \beta_{k\omega'} \delta(\omega-\omega'), \tag{8.34}$$

$$= \beta_{k\omega}. \tag{8.35}$$

(**f**) The Cartesian negative-frequency solution in Rindler coordinates reads

$$u_k = \frac{e^{i\mathfrak{a}^{-1}ke^{\mathfrak{a}\xi}(\sinh\mathfrak{a}\tau - \cosh\mathfrak{a}\tau)}}{\sqrt{4\pi k}} \tag{8.36}$$

$$= \frac{e^{-i\mathfrak{a}^{-1}ke^{-\mathfrak{a}(\tau-\xi)}}}{\sqrt{4\pi k}}. \tag{8.37}$$

Thus, their overlap with the Rindler negative-frequency solutions reads

$$(v_\omega, u_k) = \frac{1}{4\pi\sqrt{\omega k}} \int_{-\infty}^{\infty} \mathrm{d}\xi (\omega + ke^{\mathfrak{a}(\xi-\tau)}) e^{-i\left(\omega(\tau-\xi) + \frac{k}{\mathfrak{a}}e^{-\mathfrak{a}(\tau-\xi)}\right)}, \tag{8.38}$$

$$= \frac{1}{4\pi\sqrt{\omega k}} \int_{-\infty}^{\infty} \mathrm{d}V (\omega + ke^{-\mathfrak{a}V}) e^{-i\left(\omega V + \frac{k}{\mathfrak{a}}e^{-\mathfrak{a}V}\right)}, \tag{8.39}$$

where $V = \tau - \xi$. We now introduce a new variable

$$z = ke^{-\mathfrak{a}V}. \tag{8.40}$$

Then, the measure becomes

$$-\frac{\mathrm{d}z}{\mathfrak{a}z} = \mathrm{d}V. \tag{8.41}$$

Thus, we massaged the integral into the shape

$$(v_\omega, u_k) =\frac{1}{4\pi\sqrt{\omega k}}\int_0^\infty \frac{\mathrm{d}z}{\mathfrak{a}z}(\omega+z)\left(\frac{z}{k}\right)^{i\omega/\mathfrak{a}} e^{-\frac{iz}{\mathfrak{a}}}, \tag{8.42}$$

$$=\frac{k^{-\frac{i\omega}{\mathfrak{a}}}}{4\pi\sqrt{\omega k}\mathfrak{a}}\int_0^\infty \mathrm{d}z\left[z^{\frac{i\omega}{\mathfrak{a}}}\left(1+\frac{\omega}{z}\right)\right]e^{-\frac{iz}{\mathfrak{a}}}. \tag{8.43}$$

Here, we apply the definition of the Γ-function as Laplace transform of a power law

$$\int_0^\infty z^{s-1}e^{-\lambda z}\mathrm{d}z = \lambda^{-s}\Gamma(s), \tag{8.44}$$

which analytically continued to $\lambda = i/\mathfrak{a}$ and for $s = i\omega/\mathfrak{a}$ yields

$$\int_0^\infty z^{\frac{i\omega}{\mathfrak{a}}-1}e^{-\frac{iz}{\mathfrak{a}}}\mathrm{d}z = \left(\frac{i}{\mathfrak{a}}\right)^{-\frac{i\omega}{\mathfrak{a}}}\Gamma\left(\frac{i\omega}{\mathfrak{a}}\right) = \mathfrak{a}^{\frac{i\omega}{\mathfrak{a}}}e^{\frac{\pi\omega}{2\mathfrak{a}}}\Gamma\left(\frac{i\omega}{\mathfrak{a}}\right). \tag{8.45}$$

Thus, the Bogolyubov coefficient becomes

$$\alpha_{k\omega} =\frac{1}{4\pi\sqrt{\omega k}\mathfrak{a}}\mathfrak{a}^{\frac{i\omega}{\mathfrak{a}}}k^{-\frac{i\omega}{\mathfrak{a}}}e^{\frac{\pi\omega}{2\mathfrak{a}}}\left[\omega\Gamma\left(\frac{i\omega}{\mathfrak{a}}\right)-i\mathfrak{a}\Gamma\left(1+\frac{i\omega}{\mathfrak{a}}\right)\right], \tag{8.46}$$

$$=\frac{\omega}{2\pi\sqrt{\omega k}\mathfrak{a}}\mathfrak{a}^{\frac{i\omega}{\mathfrak{a}}}k^{-\frac{i\omega}{\mathfrak{a}}}e^{\frac{\pi\omega}{2\mathfrak{a}}}\Gamma\left(\frac{i\omega}{\mathfrak{a}}\right). \tag{8.47}$$

The Bogolyubov coefficient $\beta_{k\omega}$ is derived from $v^*_\omega = iv_{-\omega}$. Thus, keeping in mind that v^*_ω appears on the left entry of the Klein-Gordon inner product in Eq. (8.12) and is therefore complex conjugated, we immediately obtain

$$\beta_{k\omega} =-i\alpha_{k-\omega}, \tag{8.48}$$

$$=-\frac{\omega}{2\pi\sqrt{\omega k}\mathfrak{a}}\mathfrak{a}^{-\frac{i\omega}{\mathfrak{a}}}k^{\frac{i\omega}{\mathfrak{a}}}e^{-\frac{\pi\omega}{2\mathfrak{a}}}\Gamma\left(-\frac{i\omega}{\mathfrak{a}}\right). \tag{8.49}$$

(**g**) First we have to express the operators a_ω, $a^\dagger_\omega$ in terms of the operators b_k, $b^\dagger_k$. We find this relation by expressing

$$\phi = \int_0^\infty \mathrm{d}k\left(b_k u_k + b^\dagger_k u^*_k\right), \tag{8.50}$$

$$= \int_0^\infty \mathrm{d}k\int_0^\infty \mathrm{d}\omega\left[b_k(\alpha_{k\omega}v_\omega+\beta_{k\omega}v^*_\omega)+b^\dagger_k(\alpha^*_{k\omega}v^*_\omega+\beta^*_{k\omega}v_\omega)\right], \tag{8.51}$$

$$= \int_0^\infty \mathrm{d}\omega\int_0^\infty \mathrm{d}k\left[(\alpha_{k\omega}b_k+\beta^*_{k\omega}b^\dagger_k)v_\omega+(\beta_{k\omega}b_k+\alpha^*_{k\omega}b^\dagger_k)v^*_\omega\right]. \tag{8.52}$$

It follows that

$$a_\omega = \int_0^\infty\left(\alpha_{k\omega}b_k+\beta^*_{k\omega}b^\dagger_k\right)\mathrm{d}k. \tag{8.53}$$

Thus, we obtain for the particle-number density

$$n_\omega =V^{-1}\langle 0_{\text{inert}}|a^\dagger_\omega a_\omega|0_{\text{inert}}\rangle \tag{8.54}$$

$$=V^{-1}\int_0^\infty \mathrm{d}k\int_0^\infty \mathrm{d}k'\beta^*_{k\omega}\beta^*_{k'\omega}\langle 0_{\text{inert}}|b_k b^\dagger_{k'}|0_{\text{inert}}\rangle, \tag{8.55}$$

$$=V^{-1}\int_0^\infty \mathrm{d}k|\beta_{k\omega}|^2. \tag{8.56}$$

(**h**) We could have equivalently expressed the number density as

$$n_\omega = V^{-1}\langle 0_{\text{inert}}| -\delta(0) + a_\omega a_\omega^\dagger |0_{\text{inert}}\rangle, \tag{8.57}$$

$$= V^{-1}\left(-\delta(0) + \int_0^\infty \mathrm{d}k \int_0^\infty \mathrm{d}k' \alpha^*_{k\omega}\alpha_{k'\omega}\langle 0_{\text{inert}}|b_k b^\dagger_{k'}|0_{\text{inert}}\rangle\right), \tag{8.58}$$

$$= V^{-1}\left(-\delta(0) + \int_0^\infty \mathrm{d}k |\alpha_{k\omega}|^2\right). \tag{8.59}$$

You should be used to the pesky $\delta(0)$s now. Cancel infinity with infinity as if there was no tomorrow, wave some hands, and we obtain

$$n_\omega = -1 + V^{-1}\int_0^\infty \mathrm{d}k |\alpha_{k\omega}|^2. \tag{8.60}$$

This is fine.

We see that the Bogolyubov coefficients are related as

$$\alpha_{k\omega} = -e^{\frac{\pi\omega}{\mathfrak{a}}}\beta^*_{k\omega}. \tag{8.61}$$

Thus, we can express the particle number as

$$n_\omega = -1 + e^{\frac{2\pi\omega}{\mathfrak{a}}} V^{-1}\int_0^\infty \mathrm{d}k |\alpha_{k\omega}|^2, \tag{8.62}$$

$$= -1 + e^{\frac{2\pi\omega}{\mathfrak{a}}} n_\omega. \tag{8.63}$$

If we solve for the particle-number density, we obtain (trumpets please!)

$$n_\omega = \frac{1}{e^{\frac{2\pi\omega}{\mathfrak{a}}} - 1}. \tag{8.64}$$

This is a Planckian distribution. Thus, the field is in a thermal state – a uniformly accelerated observer is surrounded by a thermal bath of temperature $T = \mathfrak{a}/2\pi$ (in units in which $k_{\mathrm{B}} = 1$). The vacuum, thus, really has a temperature to non-inertial observers.

Extra material 1: Rindler approximation to horizons

The Rindler horizon can be understood as the first approximation of the geometry experienced by an observer hovering above a general horizon. Here's how:
Consider a general static spherically symmetric spacetime in Schwarzschild-like coordinates (t, r, ϕ, θ) such that

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right), \tag{9.1}$$

for some function $f(r)$. Let's assume that there is some $r = r_{\mathrm{h}}$ at which $f(r_{\mathrm{h}}) = 0$, $f'(r_{\mathrm{h}}) \neq 0$, implying that there is some (coordinate) singularity indicating a non-extremal horizon. Considering observers at $r > r_{\mathrm{h}}$, we can approximate the metric close to the horizon by choosing coordinates $\epsilon = r - r_{\mathrm{h}}$ and expanding in ϵ to leading order such that

$$\mathrm{d}s^2 = -f'(r_{\mathrm{h}})\epsilon \mathrm{d}t^2 + \frac{\mathrm{d}\epsilon^2}{f'(r_{\mathrm{h}})\epsilon} + r_{\mathrm{h}}^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right). \tag{9.2}$$

Next, we introduce a new radial coordinate ρ which trivializes the radial part of the metric (*i. e.* the proper distance from the horizon) such that $\mathrm{d}\rho = \mathrm{d}\epsilon/\sqrt{f'(r_{\mathrm{h}})\epsilon}$, *i. e.* $\rho = 2\sqrt{\epsilon/f'(r_{\mathrm{h}})}$. As a result, we obtain

$$\mathrm{d}s^2 = -\frac{f'(r_\mathrm{h})^2}{4}\rho^2 \mathrm{d}t^2 + \mathrm{d}\rho^2 + r_\mathrm{h}^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right). \tag{9.3}$$

Here, the (t, r)-part amounts exactly to the two-dimensional Rindler metric in terms of the proper time (in the language of exercise 7: τ) of the uniformly accelerated observer. The acceleration reads

$$\mathfrak{a} = \frac{f'(r_\mathrm{h})}{2} = \kappa, \tag{9.4}$$

where κ is usually called the surface gravity of the horizon (if gravity was still a force, this would be the force experienced on the horizon, therefore the terminology).

Schwarzschild-like time is usually defined with respect to a static observer (who is inertial only in asymptotically flat spacetimes in the limit $r \to \infty$). Here, we expanded close to the horizon, so the static observer has to be statically hovering above the horizon. This requires the radial acceleration Eq. (9.4).

In a nutshell, this means that

1. Any spacetime which is static, spherically symmetric and has a non-extremal horizon is approximated by Rindler space in the near-horizon region.

2. The Unruh effect may also apply to other kinds of horizons (chrchrm Hawking effect chrchrm). Indeed, consider the vacuum as seen by an inertial observer falling into a Schwarzschild black hole. If we plug in $f = 1 - 2M/r$, we obtain the temperature the hovering observer will experience, namely

$$T = \frac{1}{8\pi M}. \tag{9.5}$$

   We will see in a bit that this is exactly the Hawking temperature of a Schwarzschild black hole. Indeed, this is a way to derive the Hawking temperature of a general spherically symmetric black hole as

$$T = \frac{\kappa}{2\pi} = \frac{f'(r_\mathrm{h})}{4\pi}. \tag{9.6}$$

3. If there is no other length scale in the model $\kappa \propto r_\mathrm{h}^{-1}$ (amounting to wavelengths of horizon size), so the temperature is generically very low for large (e. g. astrophysical) black holes.

Exercise 9: Inflation

Motivation: In the lecture we were introduced to early-universe inflation. For those who have not seen inflation in a different course yet, this exercise should be like a very short primer. For more info, see this review.

This exercise is divided into a motivational part and a computational part.

Motivational part: Why would we want a period of accelerated expansion in the very early universe? Let's estimate. We know that the sky is made up of $N \sim 10^4$ patches which have never been in causal contact during the history of the universe if we run back ordinary FLRW evolution only with radiation and matter. If they haven't been in causal contact, we would expect them to be uncorrelated so their average temperature fluctuation $x \equiv \delta T/T_\mathrm{CMB}$ should be random, say Gaussian distributed around 0.

The important bit about the Gaussian is its standard deviation. Since x is dimensionless and there's no physical scale to set its size, a natural guess is that its standard deviation is $\Delta x \sim \mathcal{O}(1)$. In CMB measurements, correlations of x across the sky are encoded in the coefficients of the multipole expansion of the power spectrum $a_{\ell m}$. All of these coefficients have been measured to satisfy

$$|a_{\ell m}| \leq 10^{-5}. \tag{9.7}$$

Roughly, each of the first N multipole coefficients captures independent information from each causally disconnected patch. Thus, we can interpret the inequality Eq. (9.7) as measurements of x across the N independent patches.

(**a**) Compute the probability that all of the first N multipoles satisfy the inequality Eq. (9.7) if they are all Gaussian distributed around 0 with standard deviation $\Delta a_{\ell m} = 1$. You should obtain something overwhelmingly tiny.

(**b**) This has been one of the main original arguments in favour of introducing inflation: The probability of all this correlation being there randomly (the temperature fluctuations being so small everywhere) is incomprehensibly small, so the patches must have been in contact after all. Inflation brings the disconnected regions into causal contact in the past. Try to find weaknesses of this argument.

Next we get to the technical part. The usual way to go in inflation is to propose a model usually containing GR plus additional fields, which lead to accelerated expansion. This accelerated expansion implies that many areas which appear not to be causally connected now actually were during inflation. Let's start with a single minimally coupled scalar field ϕ, the inflaton – the simplest and most common type of model. The action reads

$$S = \int \mathrm{d}^x \sqrt{-g} \left(-\frac{m_\mathrm{P}^2}{2} R - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - V(\phi) \right), \tag{9.8}$$

with the Planck mass m_P, the Ricci scalar R and a potential $V(\phi)$. Inflation is usually set in an FLRW background in comoving coordinates such that

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \mathrm{d}\vec{x}^2, \tag{9.9}$$

with the scale factor $a(t)$. Then, as you have already derived, the scalar satisfies the equation of motion

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{9.10}$$

with the Hubble parameter $H = \dot{a}/a$. Besides, the cosmological principle requires the scalar, too, to only depend on time, *i. e.* $\vec{\nabla}\phi = 0$.
Slow-roll inflation occurs, when the potential energy dominates over the kinetic energy – then, the scalar does not really change and the scalar contribution to the action is approximately constant. Thus, like a cosmological constant it leads to de Sitter-like exponential expansion for some time in the very early universe. Let's make this statement more precise.

(**c**) Compute the stress energy tensor of the scalar. For a perfect fluid, the stress energy tensor equals

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu}, \tag{9.11}$$

with the energy density ρ and the pressure p and the four-velocity of the fluid $u = u^\mu \partial_\mu$ which in comoving coordinates equals $u = \partial_t$. Show that

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{9.12}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{9.13}$$

The Friedmann equations read

$$H^2 = \frac{\rho}{3m_{\mathrm{P}}^2}, \qquad \frac{\ddot{a}}{a} = -\frac{\rho + 3p}{2m_{\mathrm{P}}^2}. \tag{9.14}$$

Thus, we obtain accelerated expansion if the equation-of-state parameter of the perfect fluid w satisfies

$$w = \frac{p}{\rho} < -\frac{1}{3}. \tag{9.15}$$

Let's make precise what it means to be close to de Sitter expansion.

(d) Show that we can express the second Friedmann equation as

$$\frac{\ddot{a}}{a} = H^2(1-\epsilon), \tag{9.16}$$

where

$$\epsilon \equiv -\frac{\dot{H}}{H^2}. \tag{9.17}$$

(e) Express ϵ in terms of w to obtain

$$\epsilon = \frac{3}{2}\left(w+1\right). \tag{9.18}$$

Verify that accelerated expansion occurs for $\epsilon < 1$. (**Hint:** Perfect fluids satisfy the continuity equation $\dot{\rho} = -3H(\rho + p)$.)

The dimensionless ϵ is the first slow-roll parameter. If $\epsilon = 0$, the universe is undergoing exact de Sitter expansion (then, H is constant because $w = -1$ which amounts to a cosmological constant). So for near-de Sitter expansion, we need $\epsilon \ll 1$.

(f) Show that $\epsilon \ll 1$ implies that $\dot{\phi}^2/2 \ll V(\phi)$, *i. e.* indeed the kinetic energy is much larger than the potential energy.

Besides, for the field to roll slowly enough, the scalar should not accelerate too strongly (ϵ should not change very fast), a behaviour which is captured by the second slow-roll parameter **(In an earlier version of the assignment, the − sign was missing in the definition below.)**

$$\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}}. \tag{9.19}$$

In the slow-roll regime, $\eta \ll 1$. In the slow-roll regime, the two slow-roll parameters can be expressed in terms solely in terms of the potential and its derivatives.

(g) Demonstrate that in the slow-roll regime, **(In an earlier version of the assignment, the Planck masses were missing in the below equation)**

$$\epsilon_V \equiv \frac{m_{\mathrm{P}}^2}{2}\left(\frac{V'}{V}\right)^2 \simeq \epsilon, \tag{9.20}$$

$$\eta_V \equiv m_{\mathrm{P}}^2 \frac{V''}{V} \simeq \eta + \epsilon. \tag{9.21}$$

In the slow-roll regime, it's sufficient to monitor ϵ_V and η_V.
Inflation ends when $\epsilon = 1$. As the previous expansion was quasi exponential, it is useful to adapt our measure of time accordingly. Therefore, we introduce the "e-fold" such that

$$\mathrm{d}N = -H\mathrm{d}t, \tag{9.22}$$

normalized such that at the end of inflation $N = 0$. Thus, considering that $a \sim e^{Ht}$, at some N, the universe was smaller by a factor $\sim e^{-N}$ than at the end of inflation.
Let's get to observables. Everything that's within the horizon at the end of inflation is generically observable to us and may leave an imprint on the CMB. This amounts to fluctuations generated up to some $N = N_\star$, where depending on the model $40 \leq N_\star \leq 60$. Observables are generically evaluated at $N_\star$ because that's when the seeds of primordial fluctuations were sown. The two most well-known CMB-observables are[a]

- Δ_s^2 denotes the amplitude of spectrum of scalar fluctuations which has to be normalized as $\Delta_\mathrm{s}^2 \sim 10^{-9}$. For slow-roll inflation it reads

$$\Delta_\mathrm{s}^2 \simeq \frac{1}{24\pi^2}\frac{V}{m_\mathrm{P}^4}\epsilon_V^{-1}|_{N=N_\star}. \tag{9.23}$$

- r is the tensor-to-scalar ratio, *i. e.* the ratio of the amplitudes of tensor fluctuations (stemming from GR's spin-two part) and scalar fluctuations (stemming from the inflaton and GR's spin-zero part). Expressed in terms of the potential-derived slow-roll parameters it reads

$$r = 16\epsilon_V^\star. \tag{9.24}$$

  Hereafter the superscript $\star$ means that the respective quantity is evaluated at the effective onset of inflation $N_\star$.

- n_s is the scalar spectral index which you should have already seen in the lecture. It governs the tilt of the power spectrum of scalar fluctuations towards red. Expressed in terms of the potential-derived slow-roll parameters it reads **(In an earlier version of the assignment there was a wrong sign in the equation below.)**

$$n_\mathrm{s} = 1 + 2\eta_V^\star - 6\epsilon_V^\star. \tag{9.25}$$

Now, we compute the observables for a simple example. Consider an inflaton with quadratic potential

$$V = \frac{m^2}{2}\phi^2. \tag{9.26}$$

Assume that $N_\star = 60$, *i. e.* Inflation occurs, effectively, for 60 e-folds.

(h) Compute the value of the field at the end of inflation. From this derive $\phi_\star$, *i. e.* the field value at $N = N_\star$.

(i) Fix the mass m such that $\Delta_\mathrm{s} \sim 10^{-9}$.

(j) Compute r and n_s. Compare with constraints on r from Planck data and constraints on n_s from the latest ACT data. Is the model viable?

[a] You can find the full derivation of these quantities in the review mentioned above.

(**a**) The probability to obtain a value of 10^{-5} for the absolute value of a parameter that is Gauss distributed around 0 with standard deviation 1 reads

$$p = \frac{1}{\sqrt{2\pi}} \int_{-10^{-5}}^{10^{-5}} e^{-\frac{x^2}{2}} = \text{Erf}(10^{-5}) \sim 10^{-5}. \tag{9.27}$$

The total probability for this to happen for 10^4 parameters reads

$$p_{\text{tot}} = p^{10^4} = (10^{-5})^{10^4}, \tag{9.28}$$

which is really a remarkably small number.

(**b**) In the preceding exercise, we assumed what we would expect would be a unbiased distribution over completely uncorrelated patches in the sky. There are a number of problems with such arguments:

- **What is an unbiased distribution?** By using a Gaussian, we assume the patches to be entirely uncorrelated. But even if they had not been in causal contact in the past, they could still be correlated somehow. We've imposed a flat prior over possibilities without strong justification. In other words, a flat probability measure is still a probability measure.
- **Is this a problem of theory or initial conditions?** We can find a theory like inflation, which explains the small temperature fluctuations dynamically. Yet, the laws of physics are formulated in terms of differential equations. Differential equations propagate initial data — they don't generate it. So inflation just shifts the fine-tuning problem to earlier times.
- **Why exactly $\Delta x = 1$?** It's natural to expect dimensionless quantities to be of order 1, but that's a guess, not a rule. If we'd assumed $\Delta x \sim 10^{-11}$, the probability of the observed fluctuations would be close to one. The argument is extremely sensitive to arbitrary assumptions.
- **Why inflation?** The problem is to explain correlations between regions that seem causally disconnected. Inflation is one way to do that, but not the only way. Any mechanism that establishes correlations — even without causal contact — could in principle do the job. Given how little we know about the early universe, this argument allows for a lot of wiggle room.

(**c**) The stress energy tensor reads

$$T_{\mu\nu} \equiv -2\frac{\delta S}{\delta g^{\mu\nu}} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\rho\phi\nabla^\rho\phi - g_{\mu\nu}V. \tag{9.29}$$

As the scalar is just a function of time, we obtain

$$T_{00} = \frac{\dot{\phi}^2}{2} + V, \tag{9.30}$$

$$T_{ij} = a^2\left(\frac{\dot{\phi}^2}{2} - V\right). \tag{9.31}$$

Thus, considering that $u = \partial_t$ in comoving coordinates, we obtain the density and pressure

$$\rho = \frac{\dot{\phi}^2}{2} + V, \tag{9.32}$$

$$p = \frac{\dot{\phi}^2}{2} - V. \tag{9.33}$$

(**d**) We can write the derivative of the Hubble parameter as

$$\dot{H} = \frac{\ddot{a}}{a} - H^2. \tag{9.34}$$

Regrouping, we obtain

$$\frac{\ddot{a}}{a} = \left(\frac{\dot{H}}{H^2} + 1\right) H^2 = H^2(1-\epsilon). \tag{9.35}$$

(**e**) We need the Friedmann equations to express ϵ in terms of the equation-of-state parameter. The time derivative of the first Friedmann equation yields

$$\dot{H} = \frac{\dot{\rho}}{6m_{\mathrm{P}}^2 H}. \tag{9.36}$$

Plugging in the continuity equation

$$\dot{H} = -\frac{\rho + p}{2m_{\mathrm{P}}^2}. \tag{9.37}$$

Then, we can plug into the definition of ϵ :

$$\epsilon = \frac{3(\rho+p)}{2\rho} = \frac{3}{2}(w+1). \tag{9.38}$$

Indeed, when $\epsilon < 1$, $w < -1/3$ so the expansion is accelerated.

(**f**) According to Eq. (9.18) the regime $0 < \epsilon \ll 1$ amounts to the regime $-1 < w \ll 1/3$. Plugging in Eqs. (9.12) and (9.13), we obtain

$$-1 \leq \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V} = \frac{\frac{\frac{1}{2}\dot{\phi}^2}{V} - 1}{\frac{\frac{1}{2}\dot{\phi}^2}{V} + 1} \ll -\frac{1}{3}. \tag{9.39}$$

The value -1 is realized for $\dot{\phi} = 0$. Thus, deviations from $w = -1$ are small if

$$\frac{\frac{1}{2}\dot{\phi}^2}{V} \ll 1. \tag{9.40}$$

To make this more explicit, we can expand

$$w = -1 + \frac{\dot{\phi}^2}{V}. \tag{9.41}$$

(**g**) In the slow-roll regime, we can approximate ϵ as

$$\epsilon = \frac{3(p+\rho)}{2\rho} = \frac{3\dot{\phi}^2}{2(\frac{\dot{\phi}^2}{2} + V)} \simeq \frac{3\dot{\phi}^2}{2V}, \tag{9.42}$$

The equation of motion for the scalar approximately equals

$$3H\dot{\phi} + V' \simeq 0, \tag{9.43}$$

while the first Friedmann equation reads

$$H^2 \simeq \frac{V}{3m_{\mathrm{P}}^2}. \tag{9.44}$$

Plugging in both equations, we obtain

$$\epsilon \simeq \frac{3V'^2}{2H^2V} \simeq \frac{m_{\rm P}^2}{2}\left(\frac{V'}{V}\right)^2 \equiv \epsilon_V. \tag{9.45}$$

Additionally, the second slow-roll parameter becomes

$$\eta = -\frac{\ddot{\phi}}{H\dot{\phi}}. \tag{9.46}$$

Taking a time-derivative of the scalar-field equation, *i. e.* Eq. (9.43),

$$3\dot{H}\dot{\phi} + 3H\ddot{\phi} + V''\dot{\phi} \simeq 0. \tag{9.47}$$

Thus, we can replace $\ddot{\phi}$ to obtain

$$\eta \simeq \frac{\dot{H} + \frac{V''}{3}}{H^2} \simeq -\epsilon + m_{\rm P}^2\frac{V''}{V} \equiv -\epsilon + \eta_V. \tag{9.48}$$

Thus, we obtain

$$\eta_V = \eta + \epsilon. \tag{9.49}$$

(**h**) At the end of inflation, we have

$$\epsilon_V(\phi_{\rm end}) = \frac{m_{\rm P}^2}{2}\left(\frac{V'(\phi_{\rm end})}{V(\phi_{\rm end})}\right)^2 = \frac{2m_{\rm P}^2}{\phi_{\rm end}^2} = 1. \tag{9.50}$$

Thus, the field value at which inflation ends equals $\phi_{\rm end} = \sqrt{2}m_{\rm P}$. Using the definition of the e-fold

$$N(\phi) = \int_{t_{\rm end}}^{t} H{\rm d}t. \tag{9.51}$$

Yet, during slow-roll inflation, we can express H as a function of ϕ and change the integration to ${\rm d}t = \dot{\phi}^{-1}{\rm d}\phi \simeq -3H{\rm d}\phi/V'$.

Therefore, the integration of time becomes

$$N(\phi) = \int_{\phi_{\rm end}}^{\phi} \frac{3H^2}{V'}{\rm d}\phi \simeq \int_{\sqrt{2}m_{\rm P}}^{\phi} \frac{V}{m_{\rm P}^2 V'}{\rm d}\phi = \int_{\sqrt{2}m_{\rm P}}^{\phi} \frac{\phi}{2m_{\rm P}^2}{\rm d}\phi = \frac{1}{4}\left(\frac{\phi^2}{m_{\rm P}^2} - 2\right). \tag{9.52}$$

Thus, we can express $\phi_\star$ in terms of $N_\star$ as

$$\phi_\star = \sqrt{4N_\star + 2}m_{\rm P}. \tag{9.53}$$

At $N_\star = 60$ we obtain $\phi_\star = \sqrt{242}m_{\rm P} \simeq 16m_{\rm P}$.

(**i**) The first slow-roll parameter for the quadratic potential is given in Eq. (9.50). Thus, the amplitude of the scalar fluctuations reads

$$\Delta_{\rm s}^2 \simeq \frac{m^2\phi_\star^4}{96\pi^2 m_{\rm P}^6} = \frac{242^2}{96\pi^2}\frac{m^2}{m_{\rm P}^2}. \tag{9.54}$$

Given that $242^2/96\pi^2 \simeq 62 \sim \mathcal{O}(10)$, and $\Delta_{\rm s}^2 \sim 10^{-9}$, we obtain the mass

$$m \sim 10^{-5}m_{\rm P} \sim 10^{14}{\rm GeV}. \tag{9.55}$$

Considering that H amounts to the energy scale of inflation, and $H \sim m$, inflation occurs at around this scale.

(**j**) We can immediately compute the tensor-to-scalar ratio

$$r = 32\frac{m_{\mathrm{P}}^2}{\phi_\star^2} = \frac{32}{242} \simeq 0.1. \tag{9.56}$$

For the scalar spectral index we need the second slow-roll parameter

$$\eta_V(\phi_\star) = \frac{2m_{\mathrm{P}}^2}{\phi_\star^2} = \frac{1}{121} \simeq 0.008 = \epsilon_V(\phi_\star). \tag{9.57}$$

Thus, the scalar spectral index equals

$$n_{\mathrm{s}} \simeq 0.967. \tag{9.58}$$

Let's compare to data. The latest ACT data predicts $n_{\mathrm{s}} = 0.9709 \pm 0.0038$. Thus, the quadratic model is well within error tolerance. Error bars amount to 1σ. A little bit more than 1σ does not imply a tension. The value of r from Planck data, namely $r < 0.044$ at 3σ, cannot be accommodated in the quadratic model for inflation, however. Indeed, quadratic inflation is in 4σ-tension with the Planck data (and actually in even higher tension with newer data), and, therefore, ruled out.