Quantum field theory in curved spacetime

Assignment 5/Exam 1 – May 28

Please hand in this assignment before the tutorial at 11h15AM on May 28. In total, you need to obtain 40% of the combined points from this and the second exam.

Exercise 12: Particle creation in an expanding universe – 30pts.

Motivation: Back to square one. Let's compute the number of particles created in a more realistic scenario than before.

Consider a spatially flat universe which starts out changing adiabatically, then undergoes a rapid phase of expansion, to finally end up in another adiabatic phase. Such a universe is given by the scale factor

$$a^2(\eta) = a_1^2 + (a_2^2 - a_1^2)\frac{1+\tanh\frac{\eta}{\eta_0}}{2}, \tag{12.1}$$

with the dimensionless parameters a_1 and a_2, and the quantity η_0 which has units of time. Propagating in this universe, consider a conformally coupled, massive scalar field according to the action

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{2}\left(m^2 - \frac{R}{6}\right)\phi^2\right). \tag{12.2}$$

(a) Plot the scale factor. What do the parameters a_1, a_2 and η_0 stand for? Ignoring the flat asymptotic regions (large $|\eta|$), what part of the universe's history could the intermediate evolution be a toy model for?

(b) Show that the scale factor Eq. (12.1) results in the squared effective mass

$$m_{\text{eff}}^2 = m^2\left(a_1^2 + (a_2^2 - a_1^2)\frac{1+\tanh\frac{\eta}{\eta_0}}{2}\right) \tag{12.3}$$

for the scalar.

(c) Having applied the usual rescaling $\phi \to \chi = a\phi$, the field satisfies the usual mode equation

$$\chi_k'' + \left(k^2 + m_{\text{eff}}^2\right)\chi_k = 0. \tag{12.4}$$

Show that the mode equation can be solved by the following two linearly independent mode functions

$$v_k = \frac{e^{i\mathbf{kx} - i\omega_+\eta - i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\text{in}}}}\,{}_2F_1\left(1 + i\omega_-\eta_0, i\omega_-\eta_0; 1 - i\omega_{\text{in}}\eta_0; \frac{1+\tanh\frac{\eta}{\eta_0}}{2}\right), \tag{12.5}$$

$$u_k = \frac{e^{i\mathbf{kx} - i\omega_+\eta - i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\text{out}}}}\,{}_2F_1\left(1 + i\omega_-\eta_0, i\omega_-\eta_0; 1 + i\omega_{\text{out}}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right), \tag{12.6}$$

where we defined

$$\omega_{\text{in}}^2 \equiv k^2 + a_1^2 m^2, \qquad \omega_{\text{out}}^2 \equiv k^2 + a_2^2 m^2, \qquad \omega_\pm \equiv \frac{\omega_{\text{out}} \pm \omega_{\text{in}}}{2}, \tag{12.7}$$

and ${}_2F_1$ denotes a common type of hypergeometric function.

Hint: Try to recover the differential equation defining the hypergeometric function $f = {}_2F_1(a, b; c; z)$, namely

$$z(1-z)f'' + [c - (1+a+b)z]f' - abf = 0. \tag{12.8}$$

(**d**) Show that the mode functions asymptote to Minkowski-like positive-frequency solutions at early and late times

$$v_k \sim \frac{e^{i(\mathbf{kx} - \omega_{\rm in}\eta)}}{\sqrt{4\pi\omega_{\rm in}}}, \qquad \eta \to -\infty, \tag{12.9}$$

$$u_k \sim \frac{e^{i(\mathbf{kx} - \omega_{\rm out}\eta)}}{\sqrt{4\pi\omega_{\rm out}}}, \qquad \eta \to \infty. \tag{12.10}$$

Discuss why the mode equation has solutions with these asymptotics. Conclude that v_k defines a natural in-vacuum, and u_k a natural out-vacuum. Why?

Hint: The hypergeometric function has the limit $\lim_{z\to 0} f(a, b; c; z) = 1$ for all a, b, c.

Thus, the field can be expanded in modes as

$$\chi = \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \left(a_{\mathbf{k}} v_k + a^\dagger_{\mathbf{k}} v^*_k \right), \tag{12.11}$$

where $a_{\mathbf{k}}$ defines the in-vacuum via $a_{\mathbf{k}}|0_{\rm in}\rangle = 0$. We can write down a similar mode expansion, namely

$$\chi = \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \left(b_{\mathbf{k}} u_k + b^\dagger_{\mathbf{k}} u^*_k \right), \tag{12.12}$$

where $b_{\mathbf{k}}$ defines the out-vacuum via $b_{\mathbf{k}}|0_{\rm out}\rangle = 0$, and $b^\dagger_{\mathbf{k}}$ constructs particle states at late times. Clearly $v_k \neq u_k$. Thus, the two have to be related as

$$v_k = \alpha_k u_k + \beta_k u^*_{-k}, \tag{12.13}$$

with the Bogolyubov coefficients α_k, and β_k.

(**e**) Demonstrate that the Bogolyubov coefficients equal

$$\alpha_k = \sqrt{\frac{\omega_{\rm out}}{\omega_{\rm in}}} \frac{\Gamma(1 - i\omega_{\rm in}\eta_0)\Gamma(-i\omega_{\rm out}\eta_0)}{\Gamma(-i\omega_+\eta_0)\Gamma(1 - i\omega_+\eta_0)}, \tag{12.14}$$

$$\beta_k = \sqrt{\frac{\omega_{\rm out}}{\omega_{\rm in}}} \frac{\Gamma(1 - i\omega_{\rm in}\eta_0)\Gamma(i\omega_{\rm out}\eta_0)}{\Gamma(i\omega_-\eta_0)\Gamma(1 + i\omega_-\eta_0)}. \tag{12.15}$$

Hint: The hypergeometric function ${}_2F_1$ satisfies the identities

$$\begin{aligned} {}_2F_1(a, b; c; z) =& \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b; a+b-c+1; 1-z) \\ &+ (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} {}_2F_1(c-a, c-b; c-a-b+1; 1-z), \end{aligned} \tag{12.16}$$

$${}_2F_1(a, b; c; z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z). \tag{12.17}$$

(**f**) Assume that the field is in the vacuum state at early times. Show that the particle number density at late times equals

$$n_k = \frac{\sinh^2(\pi\omega_-\eta_0)}{\sinh(\pi\omega_{\rm in}\eta_0)\sinh(\pi\omega_{\rm out}\eta_0)}. \tag{12.18}$$

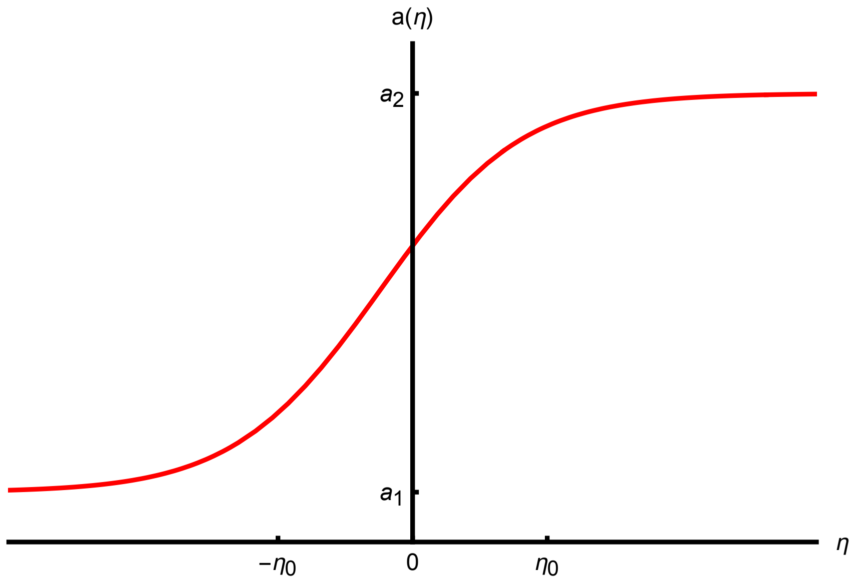

Figure 4: Scale factor given in Eq. (12.1).

(**a**) I plot the scale factor in fig. 4. It starts out flat at $a = a_1$ and ends up flat at $a = a_2$. The intermediate evolution smoothly interpolates between the two points. The parameter η_0 is the characteristic time scale over which the transition from a_1 to a_2 occurs.

To interpret the intermediate evolution, we have to keep in mind that the scale factor is given in terms of conformal time, not cosmological time. To see what is physically happening, we compute the acceleration of the scale factor in physical time t (derivatives with respect to t are denoted by overdots)

$$\ddot{a} = \frac{a''}{a^2} - \frac{a'^2}{a^3}. \tag{12.19}$$

I plot the acceleration of the scale factor during the intermediate evolution in fig. 5. In a nutshell, the intermediate evolution features accelerated expansion followed by deceleration, resembling cosmic inflation and subsequent radiation-/matter-dominated phases.

(**b**) As we computed in exercise 6, the effective mass of a nonminimally coupled scalar in an FLRW background reads

$$m_{\text{eff}}^2 = a^2 \left[m^2 + 6 \left(\xi - \frac{1}{6} \right) R \right], \tag{12.20}$$

where ξ is the nonminimal coupling. For the present case, *i. e.* $\xi = 1/6$, we obtain simply

$$m_{\text{eff}}^2 = a^2 m^2 = m^2 \left[a_1^2 + (a_2^2 - a_1^2) \frac{1 + \tanh \frac{\eta}{\eta_0}}{2} \right]. \tag{12.21}$$

(**c**) We make the ansatz

$$v_k = \frac{e^{i\mathbf{kx} - i\omega_+ \eta - i\omega_- \eta_0 \log(2\cosh \frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\text{in}}}} f(\eta). \tag{12.22}$$

As a result, f has to satisfy the differential equation

$$f'' - 2i \left(\omega_+ + \omega_- \tanh \frac{\eta}{\eta_0} \right) f' + \frac{i + \eta_0 \omega_-}{\eta_0} \frac{\omega_-}{\sinh^2 \frac{\eta}{\eta_0}} f = 0. \tag{12.23}$$

If we now introduce the new variable

$$z = \frac{1 + \tanh \frac{\eta}{\eta_0}}{2}, \tag{12.24}$$

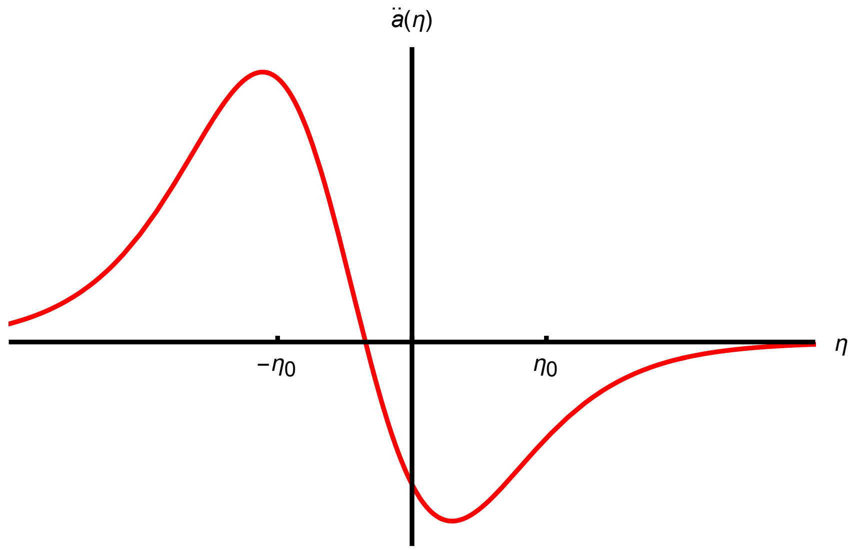

Figure 5: Scale factor acceleration with respect to cosmic time as a function of conformal time as given in Eq. (12.19).

$f(z)$ satisfies Eq. (12.8) with

$$a = 1 + i\omega_- \eta_0, \qquad b = i\omega_- \eta_0, \qquad c = 1 - i\omega_{\rm in}\eta_0. \tag{12.25}$$

Similarly, for u_k, we make the ansatz

$$u_k = \frac{e^{i\mathbf{kx} - i\omega_+ \eta - i\omega_- \eta_0 \log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\rm out}}} g(\eta), \tag{12.26}$$

yielding the differential equation

$$g'' - 2i\left(\omega_+ + \omega_- \tanh\frac{\eta}{\eta_0}\right) g' + \frac{i + \eta_0\omega_-}{\eta_0} \frac{\omega_-}{\sinh^2\frac{\eta}{\eta_0}} g = 0. \tag{12.27}$$

If we now introduce the new variable

$$w = \frac{1 - \tanh\frac{\eta}{\eta_0}}{2}, \tag{12.28}$$

$g(w)$ satisfies Eq. (12.8) with

$$a = 1 + i\omega_- \eta_0, \qquad b = i\omega_- \eta_0, \qquad c = 1 + i\omega_{\rm out}\eta_0. \tag{12.29}$$

(**d**) In the limit $\eta \to -\infty$, the variable z goes to

$$\lim_{\eta\to-\infty} z = \lim_{\eta\to-\infty} \frac{1 + \tanh\frac{\eta}{\eta_0}}{2} = 0. \tag{12.30}$$

Similarly, in the limit $\eta \to \infty$, the variable w goes to

$$\lim_{\eta\to\infty} w = \lim_{\eta\to\infty} \frac{1 - \tanh\frac{\eta}{\eta_0}}{2} = 0. \tag{12.31}$$

Then, the hypergeometric function in the mode functions v and u equals

$$_2F_1(a, b; c; 0) = 1. \tag{12.32}$$

Thus, for large negative η the mode function v_k goes like

$$v_k \sim \frac{e^{i\mathbf{kx}-i\omega_+\eta-i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\rm in}}}, \tag{12.33}$$

$$= \frac{e^{i\mathbf{kx}-i\omega_+\eta-i\omega_-\eta_0\log\left(e^{\frac{\eta}{\eta_0}}+e^{-\frac{\eta}{\eta_0}}\right)}}{\sqrt{4\pi\omega_{\rm in}}}, \tag{12.34}$$

$$\sim \frac{e^{i\mathbf{kx}-i(\omega_+-\omega_-)\eta}}{\sqrt{4\pi\omega_{\rm in}}}, \tag{12.35}$$

$$= \frac{e^{i\mathbf{kx}-i\omega_{\rm in}\eta}}{\sqrt{4\pi\omega_{\rm in}}}. \tag{12.36}$$

Similarly, for large positive η the mode function u_k goes like

$$u_k \sim \frac{e^{i\mathbf{kx}-i\omega_+\eta-i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\rm out}}}, \tag{12.37}$$

$$= \frac{e^{i\mathbf{kx}-i\omega_+\eta-i\omega_-\eta_0\log\left(e^{\frac{\eta}{\eta_0}}+e^{-\frac{\eta}{\eta_0}}\right)}}{\sqrt{4\pi\omega_{\rm out}}}, \tag{12.38}$$

$$\sim \frac{e^{i\mathbf{kx}-i(\omega_++\omega_-)\eta}}{\sqrt{4\pi\omega_{\rm out}}}, \tag{12.39}$$

$$= \frac{e^{i\mathbf{kx}-i\omega_{\rm out}\eta}}{\sqrt{4\pi\omega_{\rm out}}}. \tag{12.40}$$

The mode equation has solutions which asymptote to plane waves in Minkowski spacetime with constant scale factors a_1 as $\eta \to -\infty$ and a_2 as $\eta \to \infty$. It allows for these solutions exactly because for large negative and large positive η the spacetime itself asymptotes to Minkowski spacetime up to an overall constant rescaling by a_1 and a_2, respectively. Therefore, v is suited to describe the asymptotic Minkowski vacuum at early times, while u is suited to construct the asymptotic Minkowski vacuum at late times.

(**e**) Expressing v_k in terms of u_k by Eq. (12.13), we have

$$\begin{aligned} v_k =& e^{i\mathbf{kx}}\Bigg[\alpha_k \frac{e^{-i\omega_+\eta-i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\rm out}}}\,{}_2F_1\left(1+i\omega_-\eta_0, i\omega_-\eta_0; 1+i\omega_{\rm out}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right) \\ &+\beta_k \frac{e^{i\omega_+\eta+i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\rm out}}}\,{}_2F_1^*\left(1+i\omega_-\eta_0, i\omega_-\eta_0; 1+i\omega_{\rm out}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right)\Bigg]. \end{aligned} \tag{12.41}$$

Conjugating the differential equation defining the hypergeometric, *i. e.* Eq. (12.8) function, we obtain

$$z^*(1-z^*)f^{*\prime\prime} + [c^* - (1+a^*+b^*)z^*]f^{*\prime} - a^*b^*f^* = 0. \tag{12.42}$$

Clearly, this differential equation defines the same hypergeometric function with conjugated arguments, *i. e.*

$${}_2F_1^*(a, b; c; z) = {}_2F_1(a^*, b^*; c^*; z^*). \tag{12.43}$$

Thus, we obtain the equality

$$\begin{aligned} v_k =& e^{i\mathbf{kx}}\Bigg[\alpha_k \frac{e^{-i\omega_+\eta-i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\rm in}}}\,{}_2F_1\left(1+i\omega_-\eta_0, i\omega_-\eta_0; 1+i\omega_{\rm out}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right) \\ &+\beta_k \frac{e^{i\omega_+\eta+i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\rm in}}}\,{}_2F_1\left(1-i\omega_-\eta_0, -i\omega_-\eta_0; 1-i\omega_{\rm out}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right)\Bigg]. \end{aligned} \tag{12.44}$$

Let's now see whether we can get the left-hand side into the same shape. We use the first hint (Eq. (12.16)) to re-express the hypergeometric function in v_k as

$$v_k = \frac{e^{i\mathbf{kx}-i\omega_+\eta-i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\text{in}}}}\left[\frac{\Gamma(1-i\omega_{\text{in}}\eta_0)\Gamma(-i\omega_{\text{out}}\eta_0)}{\Gamma(-i\omega_+\eta_0)\Gamma(1-i\omega_+\eta_0)}{}_2F_1\left(1+i\omega_-\eta_0, i\omega_-\eta_0; 1+i\omega_{\text{out}}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right)\right.$$
$$\left.+\left(\frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right)^{-i\omega_{\text{out}}\eta_0}\frac{\Gamma(1-i\omega_{\text{in}}\eta_0)\Gamma(i\omega_{\text{out}}\eta_0)}{\Gamma(1+i\omega_-\eta_0)\Gamma(i\omega_-\eta_0)}{}_2F_1\left(-i\omega_+\eta_0, 1-i\omega_+\eta_0; 1-i\omega_{\text{out}}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right)\right]. \tag{12.45}$$

The first term is already in the right shape. Regarding the second term, note that we can rewrite

$$\left(\frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right)^{-i\omega_{\text{out}}\eta_0} = e^{-i\omega_{\text{out}}\eta_0\log\left(\frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right)}, \tag{12.46}$$

$$= e^{-i\omega_{\text{out}}\eta_0\log\left(\frac{e^{-\frac{\eta}{\eta_0}}}{2\cosh\frac{\eta}{\eta_0}}\right)}, \tag{12.47}$$

$$= e^{i\omega_{\text{out}}\left[\eta+\eta_0\log\left(2\cosh\frac{\eta}{\eta_0}\right)\right]}. \tag{12.48}$$

To flip the phase factor in the second term of Eq. (12.45), we reformulate

$$\left(\frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right)^{-i\omega_{\text{out}}\eta_0} = e^{2i\omega_+\eta+2i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}e^{-i(2\omega_+-\omega_{\text{out}})\eta-i(2\omega_--\omega_{\text{out}})\eta_0\log(2\cosh\frac{\eta}{\eta_0})}, \tag{12.49}$$

$$= e^{2i\omega_+\eta+2i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}e^{i\omega_{\text{in}}[-\eta+\eta_0\log(2\cosh\frac{\eta}{\eta_0})]}, \tag{12.50}$$

$$= e^{2i\omega_+\eta+2i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}e^{-i\omega_{\text{in}}\eta_0\log\left[\frac{e^{\frac{\eta}{\eta_0}}}{2\cosh\frac{\eta}{\eta_0}}\right]}, \tag{12.51}$$

$$= e^{2i\omega_+\eta+2i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}\left(\frac{1+\tanh\frac{\eta}{\eta_0}}{2}\right)^{-i\omega_{\text{in}}\eta_0}. \tag{12.52}$$

Thus, we can rewrite the second term in Eq. (12.45) as

$$v_k - \alpha_k u_k = \frac{e^{i\mathbf{kx}+i\omega_+\eta+i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\text{in}}}}\frac{\Gamma(1-i\omega_{\text{in}}\eta_0)\Gamma(i\omega_{\text{out}}\eta_0)}{\Gamma(1+i\omega_-\eta_0)\Gamma(i\omega_-\eta_0)}$$
$$\times\left(\frac{1+\tanh\frac{\eta}{\eta_0}}{2}\right)^{-i\omega_{\text{in}}\eta_0}{}_2F_1\left(-i\omega_+\eta_0, 1-i\omega_+\eta_0; 1-i\omega_{\text{out}}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right). \tag{12.53}$$

Now we use the second hint (Eq. (12.17)) to express

$$\left(\frac{1+\tanh\frac{\eta}{\eta_0}}{2}\right)^{-i\omega_{\text{in}}\eta_0}{}_2F_1\left(-i\omega_+\eta_0, 1-i\omega_+\eta_0; 1-i\omega_{\text{out}}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right) \tag{12.54}$$

$$= {}_2F_1\left(1-i\omega_-\eta_0, 1-i\omega_-\eta_0; 1-i\omega_{\text{out}}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right). \tag{12.55}$$

Thus, after all the mode function can be expressed as

$$
\begin{aligned}
v_k =& e^{i\mathbf{k}\mathbf{x}} \left[\frac{e^{-i\omega_+\eta - i\omega_-\eta_0 \log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\rm in}}} \frac{\Gamma(1-i\omega_{\rm in}\eta_0)\Gamma(-i\omega_{\rm out}\eta_0)}{\Gamma(-i\omega_+\eta_0)\Gamma(1-i\omega_+\eta_0)} {}_2F_1\left(1+i\omega_-\eta_0, i\omega_-\eta_0; 1+i\omega_{\rm out}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right)\right. \\
&\left. + \frac{e^{i\omega_+\eta + i\omega_-\eta_0 \log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\rm in}}} \frac{\Gamma(1-i\omega_{\rm in}\eta_0)\Gamma(i\omega_{\rm out}\eta_0)}{\Gamma(1+i\omega_-\eta_0)\Gamma(i\omega_-\eta_0)} {}_2F_1\left(1-i\omega_-\eta_0, -i\omega_-\eta_0; 1-i\omega_{\rm out}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right)\right].
\end{aligned}
\tag{12.56}
$$

This is exactly the shape of Eq. (12.44). Therefore, we can read off the Bogolyubov coefficients

$$
\alpha_k = \sqrt{\frac{\omega_{\rm out}}{\omega_{\rm in}}} \frac{\Gamma(1-i\omega_{\rm in}\eta_0)\Gamma(-i\omega_{\rm out}\eta_0)}{\Gamma(-i\omega_+\eta_0)\Gamma(1-i\omega_+\eta_0)}, \tag{12.57}
$$

$$
\beta_k = \sqrt{\frac{\omega_{\rm out}}{\omega_{\rm in}}} \frac{\Gamma(1-i\omega_{\rm in}\eta_0)\Gamma(i\omega_{\rm out}\eta_0)}{\Gamma(i\omega_-\eta_0)\Gamma(1+i\omega_-\eta_0)}. \tag{12.58}
$$

(**f**) As usual, we define the number density in the in-vacuum as

$$
n_{\mathbf{k}} \equiv V^{-1}\langle 0_{\rm in}| b^\dagger_{\mathbf{k}} b_{\mathbf{k}} |0_{\rm in}\rangle \tag{12.59}
$$

$$
= |\beta_k|^2, \tag{12.60}
$$

where we used a simplified version of Eq. (8.16). Plugging in Eq. (12.58), we obtain the number density

$$
n_{\mathbf{k}} = \frac{\omega_{\rm out}}{\omega_{\rm in}} \left| \frac{\Gamma(1-i\omega_{\rm in}\eta_0)\Gamma(i\omega_{\rm out}\eta_0)}{\Gamma(i\omega_-\eta_0)\Gamma(1+i\omega_-\eta_0)} \right|^2, \tag{12.61}
$$

$$
= \frac{\sinh^2(\pi\omega_-\eta_0)}{\sinh(\pi\omega_{\rm in}\eta_0)\sinh(\pi\omega_{\rm out}\eta_0)}, \tag{12.62}
$$

where we introduced the spatial volume V, and used the fact that

$$
|\Gamma(bi)|^2 = \frac{\pi}{b\sinh(\pi b)}, \qquad |\Gamma(1+bi)|^2 = \frac{\pi b}{\sinh(\pi b)}, \tag{12.63}
$$

for real b.

Exercise 13: Hawking radiation from non-singular black holes – 20pts.

Motivation: Black-hole solutions in GR are singular; it is expected that a more complete (quantum) theory of gravity can resolve this. We will consider whether regularity leaves any imprints in the Hawking temperature.

In this exercise, work with the Hayward metric. This is a metric that is not a solution to the Einstein equations (at least not for an energy-momentum tensor that satisfies the standard energy conditions). You can think of it as a phenomenological model for black holes beyond GR. The line-element in Schwarzschild-type coordinates is given by

$$
ds^2 = f(r)\,dt^2 - f(r)^{-1}dr^2 - r^2 d\Omega_2^2, \tag{13.1}
$$

with $f(r) = 1 - \frac{2G\,M\,r^2}{r^3 + 2G\,M\,\ell^2}$, with ℓ a parameter with units of length. All (non-derivative) curvature invariants remain finite in the limit $r \to 0$, as long as $\ell > 0$.

(**a**) What is the limit $\ell \to 0$?

(**b**) For spherically symmetric, static metrics in Schwarzschild-like coordinates, the event horizon is determined by the equation $g^{rr} = 0$. Find the location of the event horizon. If there is a qualitative (not just quantitative) difference to the Schwarzschild case, discuss it.

(**c**) Is the vector $\xi = \partial_t$ a Killing vector? If yes, is there a Killing horizon?

(**d**) Determine the surface gravity for a metric of the form Eq. (13.1) for an unspecified function $f(r)$; then insert $f(r)$ for the Hayward metric and discuss the difference to a Schwarzschild black hole.

(**e**) What are the implications of your previous results for the Hawking temperature of the Hayward black hole?

(**a**) In the limit $\ell \to 0$, we obtain $f = 1 - 2GM/r$, yielding the Schwarzschild metric.
(**b**) The condition $g^{rr} = f(r_{\rm H}) = 0$ amounts to the cubic equation

$$r_{\rm H}^3 + 2G\,M\,\ell^2 - 2G\,M\,r_{\rm H}^2 = 0 \tag{13.2}$$

This equation has two real, positive solutions if $\ell < 4GM/3\sqrt{3}$, namely

$$r_+ = \frac{2}{3}GM\left(1 + 2\cos\Xi\right), \qquad r_- = \frac{2}{3}GM\left(1 - \cos\Xi + \sqrt{3}\sin\Xi\right), \tag{13.3}$$

where $\Xi = \arccos(1 - 27\ell^2/8G^2M^2)/3 \leq \pi/3$ as long as $\ell \leq 4GM/3\sqrt{3}$. The two radii become equal, *i. e.* both horizons merge in the limit $\ell \to 4GM/3\sqrt{3}$. In this limit, the black hole is extremal. While we recover the single Schwarzschild horizon in the limit $\ell \to 0$, , with $\lim_{\ell\to 0} r_+ = 2GM$ and $\lim_{\ell\to 0} r_- = 0$, the Hayward black hole, thus, generally differs qualitatively from the Schwarzschild solution.

(**c**) We proved on the last exercise sheet that if the metric is independent of a coordinate, the corresponding vector is a Killing vector. The metric is independent of t, so $\xi = \partial_t$ is a Killing vector. A Killing horizon is a surface where the Killing vector becomes null. Computing the norm of ξ and setting it equal to zero, we obtain

$$\xi^2 \equiv \xi^\mu g_{\mu\nu}\xi^\nu = g_{tt} = f(r) = 0. \tag{13.4}$$

This is the same equation as the defining equation of the horizon. Thus, the horizon is a Killing horizon.

(**d**) This is analogous to exercise 10 (**d**). The surface gravity is defined such that

$$\xi^\nu \nabla_\nu \xi_\mu|_{r_\pm} = \kappa \xi_\mu|_{r=r_\pm}. \tag{13.5}$$

As Killing vector ξ satisfies the Killing equation

$$\nabla_{(\mu}\xi_{\nu)} = 0. \tag{13.6}$$

Therefore, we can express Eq. (13.5) as

$$\xi^\nu \nabla^\mu \xi_\nu|_{r=r_\pm} = \frac{1}{2}\nabla_\mu(\xi^2)|_{r=r_\pm} = -\kappa\xi_\mu|_{r=r_\pm}, \tag{13.7}$$

Projecting on some vector V which is not collinear with ξ on the horizon, we obtain

$$\kappa = -\left.\frac{V^\mu \nabla_\mu(\xi^2)}{2V^\mu \xi_\mu}\right|_{r=r_\pm}. \tag{13.8}$$

In order to be able to evaluate this equation, we need coordinates, which are well defined on the horizon, akin to the Eddington-Finkelstein coordinates in exercise 10 (**c**). Therefore, we choose

$$\mathrm{d}u = \mathrm{d}t - \mathrm{d}r_*, \tag{13.9}$$

with the tortoise coordinate satisfying $\mathrm{d}r_* = \mathrm{d}r/f(r)$. Replacing the time coordinate with the light-like coordinate u, the metric reads

$$\mathrm{d}s^2 = f\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2\mathrm{d}\Omega^2. \tag{13.10}$$

Now we can contract with $V = \partial_r$ to obtain

$$\kappa = -\left.\frac{V^\mu \nabla_\mu(\xi^2)}{2V^\mu \xi_\mu}\right|_{r=r_\pm} = \frac{f'(r_\pm)}{2}. \tag{13.11}$$

For the Hayward black hole, we obtain

$$\kappa = GMr_\pm \frac{r_\pm^3 - 4GM\ell^2}{(2GM\ell^2 + r_\pm^3)^2} = \frac{3}{4GM} - \frac{1}{r_\pm}, \tag{13.12}$$

where in the last equality we used Eq. (13.2). At the outer horizon ($r = r_+$), the surface gravity reads explicitly

$$\kappa = \frac{3}{4GM}\left(1 - \frac{2}{1 + 2\cos\Xi}\right) \le \frac{1}{4GM}, \tag{13.13}$$

where the inequality holds for all $\ell \le 4GM/3\sqrt{3}$, and κ_{S} denotes the surface gravity of the horizon of a Schwarzschild black hole, which is, of course, recovered for $\ell \to 0$. Thus, the surface gravity of the Hayward black hole is lower than that of the Schwarzschild black hole. In total, it interpolates between the Schwarzschild value $\kappa|_{\ell=0} = \kappa_{\mathrm{S}}$ and $\kappa|_{\ell=4GM/3\sqrt{3}} = 0$. Thus, in the extremal limit the temperature vanishes, just as it did for the Kerr black hole last week.

(**e**) The Hawking temperature of the outer horizon of the Hayward black hole reads

$$T = \frac{\kappa}{2\pi} = \frac{3}{8\pi GM} - \frac{1}{2\pi r_\pm} = \frac{3}{8\pi GM}\left(1 - \frac{2}{1 + 2\cos\Xi}\right) \le \frac{1}{8\pi GM} = T_{\mathrm{S}}, \tag{13.14}$$

where T_{S} denotes the temperature of the Schwarzschild black hole. Thus, the temperature is lower than that of the Schwarzschild black hole, and vanishes in the extremal limit, just like for the Kerr black hole. This has major implications for black hole evaporation: Black hole evaporation is dominated by the Stefan-Boltzmann law

$$\frac{\mathrm{d}E}{\mathrm{d}t} \propto T^4, \tag{13.15}$$

where E is the radiated energy. The Schwarzschild black hole increases its temperature while evaporating, which further increases the energy loss due to evaporation. Hence, a Schwarzschild black hole enters a vicious circle which only ends when the black hole has fully evaporated which is in finite time. Instead, the Hayward metric gets closer and closer to extremality, effectively lowering its temperature once it is small enough. Thus, when the horizon radius of the Hayward black hole becomes comparable to the regulator scale ℓ, its evaporation will slow down drastically. As it is impossible to lower the temperature of any object to zero by a finite number of steps, the Hayward black hole will never evaporate fully, but slowly enter an adiabatic stage, becoming a remnant. Note, though, that any physics at the final stages of Hawking evaporation hinges on concepts derived from QFT in curved spacetime, which is not a good approximation any more – at the final stages, quantum gravity cannot be neglected.