Quantum field theory in curved spacetime

Assignment 6 – June 4

Exercise 14: Superradiance

Motivation: Two weeks ago, we learned about particles in the ergoregion. These effects get supercharged once we talk about fields. This will be an explosive adventure!

Superradiance amounts to extraction of energy from a black hole by perturbing the Kerr geometry. First, let's find a thermodynamic argument that makes superradiance plausible in GR. In the last weeks, we have often encountered the laws of black hole mechanics/thermodynamics. Here, we need the first law

$$\delta M = \frac{\kappa}{8\pi}\delta A_{\mathrm{H}} + \Omega_{\mathrm{H}}\delta J, \tag{14.1}$$

with the change in the area δA_{H}, the surface gravity κ, the angular velocity of the horizon Ω_{H} and the change in the black hole's angular momentum δJ. Besides, we need the second law $\delta A_{\mathrm{H}} \geq 0$. The perturbation of angular momentum and energy by an incident wave of frequency ω and azimuthal number m reads

$$\frac{\delta J}{\delta M} = \frac{m}{\omega}. \tag{14.2}$$

(a) Compute the frequency range, within which we can extract mass from the black hole.

Next, we compute this explicitly for a scalar test field. The Klein-Gordon equation for a minimally coupled massless scalar ϕ on the Kerr geometry is a complicated beast. But there is a surprising extra piece of information, which allows to simplify it enormously.

(b) Consider the tensor

$$K_{\mu\nu} = r^2 g_{\mu\nu} - 2\rho^2 l_{(\mu} n_{\nu)}, \tag{14.3}$$

where l and n are null vectors satisfying $l^\mu n_\mu = 1$. In Boyer-Lindquist coordinates

$$l = \frac{r^2+a^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\phi, \qquad n = \frac{r^2+a^2}{2\rho^2}\partial_t - \frac{\Delta}{2\rho^2}\partial_r + \frac{a}{2\rho^2}\partial_\phi. \tag{14.4}$$

Using your favourite symbolic equation manipulation software, verify that K satisfies the generalized Killing equation

$$\nabla_{(\rho} K_{\mu\nu)} = 0 \tag{14.5}$$

on the Kerr geometry. Such a tensor is called a Killing tensor.

Hint: Do not do this computation by hand unless you really like to grind.

(c) Consider a general Killing tensor $k_{\mu_1\mu_2\dots\mu_n}$, satisfying the generalization of Eq. (14.6)

$$\nabla_{(\rho} k_{\mu_1\mu_2\dots\mu_n)} = 0. \tag{14.6}$$

Show that given the momentum $p^\mu = m u^\mu$, with mass m and four-velocity u^μ, one can construct a scalar $s = k_{\mu_1\mu_2\dots\mu_n} p^{\mu_1} p^{\mu_2} \dots p^{\mu_n}$, which is conserved on geodesics.

Thus, the Kerr geometry has a hidden symmetry. This symmetry is instrumental in solving the Klein-Gordon equation because it allows to construct a hermitian derivative operator which commutes with the d'Alembertian, namely $\nabla_\mu K^{\mu\nu}\nabla_\nu$.[a] As a result, one can simultaneously diagonalize the operators $\Box$, $K^\mu\nabla_\mu$, $R^\mu\nabla_\mu$ (here we use the notation of exercise 11) and $\nabla_\mu K^{\mu\nu}\nabla_\nu$. Therefore, it does not come as a surprise that the Klein-Gordon equation on Kerr can be brought into the form

$$\begin{aligned}&\left[\frac{(r^2+a^2)^2}{\Delta}-a^2\sin^2\theta\right]\partial_t^2\Psi+\frac{4GMar}{\Delta}\partial_t\partial_\phi\Psi\\&+\left[\frac{a^2}{\Delta}-\frac{1}{\sin^2\theta}\right]\partial_\phi^2\Psi-\partial_r\left(\Delta\partial_r\Psi\right)-\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\Psi)=0.\end{aligned} \tag{14.7}$$

(d) Show that Eq. (14.7) is separable: Making the ansatz

$$\Psi=\frac{1}{2\pi}\int \mathrm{d}\omega e^{-i\omega t}e^{im\phi}S(\theta)R(r) \tag{14.8}$$

show that the radial part satisfies the scalar Teukolsky equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\bar{V}R=0, \tag{14.9}$$

with

$$\bar{V}=\frac{\left[\omega(r^2+a^2)-am\right]^2}{\Delta}-a^2\omega^2+2am\omega-A_{\ell m}, \tag{14.10}$$

and where $A_{\ell m}$ are the eigenvalues to the equation

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}S}{\mathrm{d}\theta}\right)+\left(a^2\omega^2\cos^2\theta-\frac{m^2}{\sin^2\theta}+A_{\ell m}\right)S=0. \tag{14.11}$$

For example, for small $a\omega$ the eigenvalue becomes $A_{\ell m}=\ell(\ell+1)$.

Hint: In order to get the conventional shape of $A_{\ell m}$ we aim at here, you may transform θ-dependent parts in Eq. (14.7) into combinations of constant and θ-dependent terms using trigonometric identities, and shift the constant contributions into the radial differential equation.

(e) Introduce the tortoise coordinate we already know from earlier assignments

$$\mathrm{d}r_\star=\frac{r^2+a^2}{\Delta}\mathrm{d}r. \tag{14.12}$$

Recall for later that $r_\star\to-\infty$ as $r\to r_+$, while $r_\star\to\infty$ as $r\to\infty$. Find a redefinition $R(r_\star)\to\psi(r_\star)$ such that Eq. (14.9) becomes a Schrödinger-like differential equation

$$\psi''(r_\star)+V_{\text{eff}}\psi(r_\star)=0, \tag{14.13}$$

with the effective potential

$$V_{\text{eff}}=\frac{\Delta}{(r^2+a^2)^4}\left[(2r^2-a^2)\Delta-2r^2(r^2+a^2)+(r^2+a^2)^2\bar{V}\right]. \tag{14.14}$$

Consider a scattering experiment with a monochromatic wave. The boundary conditions for such an experiment have to be set at the outer horizon $r \to r_+$, and at infinity, *i. e.* $r \to \infty$. At infinity, there is an incoming wave with amplitude $\mathcal{I}$ and a reflected wave with amplitude $\mathcal{R}$. At the horizon, there is only a transmitted wave, *i. e.* one that enters the horizon with amplitude $\mathcal{T}$ because the horizon is a one-way surface.

(f) Show that Eq. (14.13) at the boundaries ($r \to r_+$, *i. e.* $r_\star \to -\infty$ and $r \to \infty$, *i. e.* $r_\star \to \infty$) allows for the solutions

$$\psi \sim \begin{cases} \mathcal{T} e^{-ik_\mathrm{H} r_\star} & r_\star \to -\infty, \\ \mathcal{I} e^{-i\omega r_\star} + \mathcal{R} e^{i\omega r_\star} & r_\star \to \infty, \end{cases} \tag{14.15}$$

where $k_\mathrm{H} = \omega - m\Omega_\mathrm{H}$ with, again, the angular velocity of the horizon $\Omega_\mathrm{H} = a/2GMr_+$.

(g) As the potential is real, the complex conjugate of Eq. (14.30) is a linearly independent solution of the equation of motion. Then, the Wronskian of ψ is independent of $r_\star$. Use this fact to compute that

$$|\mathcal{R}|^2 = |\mathcal{I}|^2 - \frac{k_\mathrm{H}}{\omega}|\mathcal{T}|^2. \tag{14.16}$$

(h) The energy flux at infinity reads

$$\frac{\mathrm{d}E_\mathrm{out}}{\mathrm{d}t} = \frac{\omega^2}{2}|\mathcal{R}|^2, \qquad \frac{\mathrm{d}E_\mathrm{in}}{\mathrm{d}t} = \frac{\omega^2}{2}|\mathcal{I}|^2, \tag{14.17}$$

where E_out and E_in stand for ingoing and outgoing energy. What happens if $\omega < m\Omega_\mathrm{H}$?

(i) **Bonus Sci-fi-question:** What happens if we surround the black hole by a perfectly reflecting (and extremely durable) mirror and send in an initial wave of frequency $\omega < \Omega_\mathrm{H}$? Not without reason, this is called a black-hole bomb.

[a] Note that K being a Killing tensor is a necessary but not a sufficient condition for $\nabla_\mu K^{\mu\nu}\nabla_\nu$ to commute with the d'Alembertian. It also has to be compatible with the curvature on the geometry, which K on Kerr actually is.

(a) Plugging the angular momentum into the first law of black hole mechanics

$$\delta M = \frac{\kappa}{8\pi}\frac{\delta A_\mathrm{H}}{1 - m\frac{\Omega_\mathrm{H}}{\omega}}. \tag{14.18}$$

Mass is extracted if $\delta M < 0$. As $\delta A_\mathrm{H} \geq 0$, mass extraction therefore requires $\omega < m\Omega_\mathrm{H}$.

(b) See ancillary Mathematica notebook.

(c) The time derivative along a geodesic equals $\mathrm{d}/\mathrm{d}\tau = u^\mu \nabla_\mu$ (here τ is the affine parameter along the curve). Thus, the time derivative of the scalar reads

$$\frac{\mathrm{d}k_{\mu_1\mu_2\ldots\mu_n}p^{\mu_1}p^{\mu_2}\ldots p^{\mu_n}}{\mathrm{d}\tau} = m^{-1}\nabla_{(\nu}k_{\mu_1\mu_2\ldots\mu_n)}p^\nu p^{\mu_1}p^{\mu_2}\ldots p^{\mu_n} + nk_{(\mu_1\mu_2\ldots\mu_n)}\frac{\mathrm{d}p^{\mu_1}}{\mathrm{d}\tau}p^{\mu_2}\ldots p^{\mu_n} = 0, \tag{14.19}$$

where in the last equality we used Eq. (14.6) and that test particles on geodesics follow unaccelerated motion, *i. e.* $\mathrm{d}p^\mu/\mathrm{d}\tau = 0$.

(**d**) Plugging in the ansatz, Eq. (14.8), we obtain

$$\begin{aligned}
&-\omega^2\left[\frac{(r^2+a^2)^2}{\Delta}-a^2\sin^2\theta\right]+\frac{4GMar}{\Delta}m\omega \\
&-\left[\frac{a^2}{\Delta}-\frac{1}{\sin^2\theta}\right]m^2-R^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta\frac{\mathrm{d}R}{\mathrm{d}r}\right)-\frac{1}{S\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}S}{\mathrm{d}\theta}\right)=0.
\end{aligned} \tag{14.20}$$

Observe that terms depending on θ and r appear separately, never mixed. Hence, we can separate these variables: The contributions to the Klein-Gordon equation dependent on r have to be constant and so have to be the contributions dependent on θ to cancel each other for all values of the variables. Thus, we set the contributions dependent on r equal to the constant $A_{\ell m}$. Then, those contributions dependent on θ have to be equal to $-A_{\ell m}$ for the Klein-Gordon equation to be satisfied.

As suggested in the hint, we write $\sin^2\theta = 1-\cos^2\theta$ in the first line and shift the constant bit to the differential equation for R. Thus, for R, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\left[\omega^2\frac{(r^2+a^2)^2}{\Delta}+\frac{a^2m^2}{\Delta}-a^2\omega^2-\frac{4GMar}{\Delta}m\omega-A_{\ell m}\right]R=0. \tag{14.21}$$

We can rewrite this equation as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\left[\frac{\left[(r^2+a^2)\omega-am\right]^2}{\Delta}+2am\omega\frac{r^2-2GMr+a^2}{\Delta}-a^2\omega^2-A_{\ell m}\right]R=0. \tag{14.22}$$

Plugging in the definition of Δ, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\left[\frac{\left[(r^2+a^2)\omega-am\right]^2}{\Delta}+2am\omega-a^2\omega^2-A_{\ell m}\right]R=0. \tag{14.23}$$

Apart from that, for the differential equation for S we obtain

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}S}{\mathrm{d}\theta}\right)+\left(a^2\omega^2\cos^2\theta-\frac{m^2}{\sin^2\theta}+A_{\ell m}\right)S=0. \tag{14.24}$$

(**e**) In terms of the tortoise coordinate, the derivative part of Eq. (14.9) reads

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta\frac{\mathrm{d}R}{\mathrm{d}r}\right)&=\frac{r^2+a^2}{\Delta}\frac{\mathrm{d}}{\mathrm{d}r_\star}\left((r^2+a^2)\frac{\mathrm{d}R}{\mathrm{d}r_\star}\right), &(14.25)\\
&=\frac{(r^2+a^2)^2}{\Delta}\frac{\mathrm{d}^2R}{\mathrm{d}r_\star^2}+2r\frac{\mathrm{d}R}{\mathrm{d}r_\star}. &(14.26)
\end{aligned}$$

To obtain a Schrödinger-like equation, we have to eliminate the first-derivative term. Indeed, if we redefine $\psi\equiv\sqrt{r^2+a^2}R$, we obtain

$$\frac{(r^2+a^2)^2}{\Delta}\frac{\mathrm{d}^2R}{\mathrm{d}r_\star^2}+2r\frac{\mathrm{d}R}{\mathrm{d}r_\star}=\frac{(r^2+a^2)^{3/2}}{\Delta}\psi''+\frac{(2r^2-a^2)\Delta-2r^2(r^2+a^2)}{(a^2+r^2)^{5/2}}\psi. \tag{14.27}$$

The Schrödinger-like equation Eq. (14.13) follows immediately.

(**f**) At the horizon $\Delta=0$, so terms proportional to Δ can be neglected. The surviving terms in the effective potential given in Eq. (14.14) read

$$V_{\text{eff}}|_{r=r_+}=\left(\omega-\frac{am}{r_+^2+a^2}\right)^2=(\omega-m\Omega_{\mathrm{H}})^2=k_{\mathrm{H}}^2. \tag{14.28}$$

Thus, as $r \to r_+$ the field ψ behaves like a plane wave with frequency k_{H}.

At large r, we can neglect terms at higher than zeroth order in $1/r$, and we have $\Delta \simeq r^2$. Thus, the effective potential reads

$$V_{\text{eff}} = \omega^2, \tag{14.29}$$

implying plane wave behaviour at infinity with frequency ω.

Thus assuming that the horizon is a one-way surface, we obtain

$$\psi \sim \begin{cases} \mathcal{T} e^{-ik_{\mathrm{H}} r_\star} & r_\star \to -\infty, \\ \mathcal{I} e^{-i\omega r_\star} + \mathcal{R} e^{i\omega r_\star} & r_\star \to \infty. \end{cases} \tag{14.30}$$

(**g**) The Wronskian of ψ and ψ^* reads

$$W(\psi, \psi^*) = \psi \frac{\mathrm{d}\psi^*}{\mathrm{d}r_\star} - \psi^* \frac{\mathrm{d}\psi}{\mathrm{d}r_\star} \tag{14.31}$$

On the horizon, the Wronskian becomes

$$W(\psi, \psi^*) = 2|\mathcal{T}|^2 i k_{\mathrm{H}}, \qquad r_\star \to -\infty, \tag{14.32}$$

while at large $r_\star$, we obtain

$$W(\psi, \psi^*) = -2i|\mathcal{R}|^2 i\omega + 2i\omega|\mathcal{I}|^2, \qquad r_\star \to \infty. \tag{14.33}$$

Thus, constancy of the Wronskian imposes the condition

$$|\mathcal{R}|^2 = |\mathcal{I}|^2 - \frac{k_{\mathrm{H}}}{\omega}|\mathcal{T}|^2. \tag{14.34}$$

(**h**) Using Eq. (14.16), the outgoing energy flux satisfies

$$\frac{\mathrm{d}E_{\text{out}}}{\mathrm{d}t} = \frac{\omega^2}{2}\left(|\mathcal{I}|^2 - \frac{k_{\mathrm{H}}}{\omega}|\mathcal{T}|^2\right) = \frac{\mathrm{d}E_{\text{in}}}{\mathrm{d}t} - \frac{k_{\mathrm{H}}\omega}{2}|\mathcal{T}|^2. \tag{14.35}$$

If $\omega < m\Omega_{\mathrm{H}}$, we have $k_{\mathrm{H}} < 0$. This immediately implies that

$$\frac{\mathrm{d}E_{\text{out}}}{\mathrm{d}t} > \frac{\mathrm{d}E_{\text{in}}}{\mathrm{d}t}, \tag{14.36}$$

i. e. net energy is extracted from the black hole.

(**i**) As the outgoing wave from the black hole has the same frequency as the incident wave, when reflected back by the mirror it will produce another, further amplified outgoing wave and so on. Like the Penrose process, this process decreases the angular momentum of the black hole, *i. e.* it decreases the angular velocity of the horizon Ω_{H}. Thus, it will do so until $\omega = m\Omega_{\mathrm{H}}$, where the energy extraction stops, and the whole system equilibrates. As we can, in principle, choose ω to be arbitrarily small (which requires making the mirror arbitrarily large), similarly to the Penrose process, one can basically reduce the angular momentum of the black hole to zero this way, releasing the same amount of energy (29% of the initial black-hole mass). Now imagine, the mirror breaks or is made to disappear at the very end of the process. This would release all of this energy at once, *i. e.* for Sagittarius A* if it was extremal $\sim 0.25\%$ of all energy that all stars in the galaxy will emit during their whole lifetime. Sounds like a galaxy-sized nuke, doesn't it?

Exercise 15: Superradiance in analogue systems

Motivation: Last time I checked, we couldn't send waves to black holes and pick up the amplified reflected waves. But we can see the effect already in a draining bath tub. Here's how.

The idea of analogue gravity is to study the effects of fields on curved spacetime in more accessible systems realizable in a lab. An example is water going down a drain. Similar to a black hole, the speed required to avoid falling down the drain exceeds the sound speed of the liquid such that perturbations experience an effective horizon. Here we try to derive the effective curved-spacetime Klein-Gordon equation these perturbations satisfy.

We describe the fluid by a velocity potential Φ in terms of which one defines the velocity field $\vec{v} = -\vec{\nabla}\Phi$. Besides, the fluid has vanishing viscosity and a barotropic equation of state $p(\rho)$ relating pressure p and density ρ, and the evolution is adiabatic, *i. e.* slow enough for the system to remain in local equilibrium. Then, the fluid satisfies the continuity equation

$$\partial_t \rho + \vec{\nabla}(\rho \vec{v}) = 0 \tag{15.1}$$

and, absent external driving forces, the Euler equation

$$-\partial_t \Phi + \frac{v^2}{2} + \int \frac{\mathrm{d}p}{\rho} = 0. \tag{15.2}$$

(a) Assume $\rho = \rho_0 + \epsilon\rho_1$, $\vec{v} = \vec{v}_0 - \epsilon\vec{\nabla}\phi$, $\Phi = \Phi_0 + \epsilon\phi$, where $\epsilon \ll 1$. Obtain the linearized continuity equation

$$\partial_t \rho_1 + \vec{\nabla}(\rho_1 \vec{v}_0 - \rho_0 \vec{\nabla}\phi) = 0, \tag{15.3}$$

and the linearized Euler equation

$$-\partial_t \phi - \vec{v}_0 \cdot \vec{\nabla}\phi + c_\mathrm{s}^2 \frac{\rho_1}{\rho_0} = 0, \tag{15.4}$$

with the background speed of sound $c_\mathrm{s}^2 = \mathrm{d}p/\mathrm{d}\rho|_{\rho=\rho_0}$.

(b) Use Eqs. (15.3) and (15.4) to obtain the differential equation for the perturbation

$$\partial_t \left(c_\mathrm{s}^{-2} \rho_0 (\partial_t \phi + \vec{v}_0 \cdot \vec{\nabla}\phi) \right) - \vec{\nabla} \cdot \left[\rho_0 \vec{\nabla}\phi - c_\mathrm{s}^{-2} \rho_0 \vec{v}_0 (\partial_t \phi + \vec{v}_0 \cdot \vec{\nabla}\phi) \right] = 0. \tag{15.5}$$

(c) Show that this equation can be equivalently obtained as the curved-space Klein-Gordon equation derived from the metric

$$\mathrm{d}s^2 = \frac{\rho_0}{c_\mathrm{s}} \left[c_\mathrm{s}^2 \mathrm{d}t^2 - (\mathrm{d}\vec{x} - \vec{v}_0 \mathrm{d}t)^2 \right]. \tag{15.6}$$

(d) As an example, consider a fluid in $2+1$ dimensions with constant c_s and a background-fluid velocity (in terms of polar coordinates r and ϕ)

$$\vec{v}_0 = -\frac{A}{r}\vec{e}_r + \frac{B}{r}\vec{e}_\phi, \tag{15.7}$$

where A, B are constants characterizing the direction of the flow. Here, we assume that $A, B > 0$ such that the fluid is moving inward with an additional clockwise rotation, thus mimicking a rotating black hole. For example, water in a draining bath tub flows this way.

Show that the resulting metric reads

$$\mathrm{d}s^2 = c_\mathrm{s}^2 \mathrm{d}t^2 - \left(\mathrm{d}r + \frac{A}{r}\mathrm{d}t \right)^2 - \left(r\mathrm{d}\phi - \frac{B}{r}\mathrm{d}t \right)^2. \tag{15.8}$$

(**e**) Show that the resulting geometry has an ergosphere at radii below

$$r_{\partial \mathrm{E}} = \frac{\sqrt{A^2+B^2}}{c_\mathrm{s}}, \tag{15.9}$$

and a sound horizon at

$$r_\mathrm{H} = \frac{A}{c_\mathrm{s}}. \tag{15.10}$$

(**f**) There is an ergoregion outside the sound horizon. What happens if you introduce a perturbation of sufficiently low frequency at the outer boundary of the experiment? What could the "sufficiently low frequency" be precisely? Think about your answer, then check out this link.

(**a**) At first order in ϵ, the linearized continuity equation, Eq. (15.3), follows immediately from Eq. (15.1). To obtain Eq. (15.4), we need to take a closer look at the specific enthalpy

$$h(p) \equiv \int \frac{\mathrm{d}p}{\rho(p)} = h(p_0+\epsilon p_1) \simeq h(p_0) + \epsilon p_1 h'(p_0) = h(p_0) + \epsilon \frac{p_1}{\rho_0}, \tag{15.11}$$

where $\rho_0 = \rho(p_0)$. At the same time, Taylor expanding $\rho(p)$ fixes $p_1 = c_\mathrm{s}^2 \rho_1$, where $c_\mathrm{s}^2 = \mathrm{d}p/\mathrm{d}\rho|_{\rho_0}$. Therefore, we obtain

$$h \simeq h(p_0) + \epsilon \frac{c_\mathrm{s}^2 \rho_1}{\rho_0}. \tag{15.12}$$

This immediately implies Eq. (15.4).

(**b**) We first solve Eq. (15.4) such that

$$\rho_1 = \frac{\rho_0}{c_\mathrm{s}^2}\left(\partial_t \phi + \vec{v}_0 \cdot \vec{\nabla}\phi\right). \tag{15.13}$$

Plugging into Eq. (15.3), we obtain Eq. (15.5).

(**c**) The inverse of the metric has the components

$$g^{tt} = \frac{1}{c_\mathrm{s}\rho_0}, \qquad g^{it} = \frac{v_0^i}{c_\mathrm{s}\rho_0}, \qquad g^{ij} = \frac{v_0^i v_0^j - c_\mathrm{s}^2 \delta^{ij}}{c_\mathrm{s}\rho_0}. \tag{15.14}$$

The determinant of the metric equals $g = -\rho_0^4/c_\mathrm{s}^2$. Thus, the d'Alembertian reads

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right), \tag{15.15}$$

$$= \frac{c_\mathrm{s}}{\rho_0^2}\partial_t\left[\frac{\rho_0}{c_\mathrm{s}^2}\left(\partial_t\phi + \vec{v}_0\cdot\vec{\nabla}\phi\right)\right] + \frac{c_\mathrm{s}}{\rho_0^2}\vec{\nabla}\cdot\left[\frac{\rho_0}{c_\mathrm{s}^2}\left(\vec{v}_0(\partial_t\phi + \vec{v}_0\cdot\vec{\nabla}\phi) - c_\mathrm{s}^2\vec{\nabla}\phi\right)\right]. \tag{15.16}$$

Finally, the Klein-Gordon equation $\Box\phi = 0$ can be expressed as

$$\partial_t\left[c_\mathrm{s}^{-2}\rho_0\left(\partial_t\phi + \vec{v}_0\cdot\vec{\nabla}\phi\right)\right] - \vec{\nabla}\cdot\left[\rho_0\vec{\nabla}\phi - c_\mathrm{s}^{-2}\rho_0\vec{v}_0\left(\partial_t\phi + \vec{v}_0\cdot\vec{\nabla}\phi\right)\right], \tag{15.17}$$

which is exactly Eq. (15.5).

(**d**) To get to polar coordinates, we transform $x^1 = r\cos\phi$, $x^2 = r\sin\phi$. Writing $\vec{v}_0 = v_0^r\vec{e}_r + v_0^\phi\vec{e}_\phi$, we can express the metric as

$$\mathrm{d}s^2 = \frac{\rho_0}{c_\mathrm{s}}\left[c_\mathrm{s}^2\mathrm{d}t^2 - \left(\mathrm{d}r^2 - v_0^r\mathrm{d}t\right)^2 - (r\mathrm{d}\phi - v_0^\phi\mathrm{d}t)^2\right]. \tag{15.18}$$

Plugging in for v_0^r and v_0^ϕ, we obtain Eq. (15.8).

(**e**) An ergosphere is a region where the timelike Killing vector becomes spacelike. The boundary of the ergosphere is defined where the norm of the timelike Killing vector becomes zero. Thus, at $r = r_{\partial\mathrm{E}}$, the vector $K = \partial_t$ is null. In other words

$$g_{tt}|_{r=r_{\partial\mathrm{E}}} = \frac{\rho_0}{c_\mathrm{s}}\left[c_\mathrm{s}^2 - \left(\frac{A}{r_{\partial\mathrm{E}}}\right)^2 - \left(\frac{B}{r_{\partial\mathrm{E}}}\right)^2\right] = 0. \tag{15.19}$$

This equation has one positive solution namely Eq. (15.9).

At the horizon, the absolute value of the radial velocity is as large as the speed of sound, similarly to the escape velocity on the horizon of a black hole being the speed of light. Thus, we set

$$|v_0^r|_{r=r_\mathrm{H}} = \frac{A}{r_\mathrm{H}} = c_\mathrm{s}, \tag{15.20}$$

which immediately implies Eq. (15.10).

(**f**) Provided the frequency is small enough, the wave is going to get amplified when it is reflected. Thus, there is going to be superradiance exactly as we derived it in the preceding exercise. The paper linked on the sheet is the first observation of this process in a lab, indeed done with a draining bath tub.

To find out what the "sufficiently low frequency" is, we have to compute the angular velocity at the horizon Ω_H. Since the angular motion of the fluid defines the frame-dragging at the horizon, Ω_H is simply the angular velocity of the fluid at the horizon, *i. e.*

$$\Omega_\mathrm{H} = \frac{v_\phi}{r}\bigg|_{r=r_\mathrm{H}} = \frac{B}{A^2}c_\mathrm{s}^2. \tag{15.21}$$

By analogy with the result in exercise 14, low frequencies are now $\omega < mBc_\mathrm{s}^2/A^2$, where m denotes the azimuthal wave number.