Quantum field theory in curved spacetime

Assignment 7 – June 16

Exercise 16: Euler-Heisenberg Lagrangian

Motivation: In semi-classical gravity, one tries to also compute the backreaction of the quantum fields on the geometry. Here, we use QED as a toy model for this kind of computation, treating the electromagnetic field as a classical background, just as one does in QFT in curved spacetime. If you think Schwinger, you're exactly right (this reference may be more understandable).

Note: This sheet is a longer one, but every single sub-exercise is doable with the hints that are given. The derivation is pretty technical. If you want to skip a step, the result of the sub-exercise is usually provided, so you can continue on with the next sub-exercise. The physical interpretation of what we compute here is in sub-exercises (**l**) and (**m**). So if you're first and foremost interested in exploring the physics, concentrate on those.

The QED partition function on a flat background reads

$$Z = \int \mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS_{\text{QED}}}, \tag{16.1}$$

with the QED action

$$S_{\text{QED}} = \int_x \left[-\mathcal{F} + \bar{\psi}\left(i\not{D} - m\right)\psi\right]. \tag{16.2}$$

Here, $\mathcal{F} \equiv F_{\mu\nu}F^{\mu\nu}/4$, where $F_{\mu\nu}$ is the field-strength tensor of the gauge field A_μ. Besides, $\not{D}$ denotes the covariant Dirac operator involving the covariant derivative

$$D_\mu = \partial_\mu + ieA_\mu, \tag{16.3}$$

and e and m are the charge and the mass of the (Grassmann-valued) fermion ψ, respectively. We want to compute the one-loop effective Lagrangian for constant $F_{\mu\nu}$ by integrating out the fermion. For constant field strength, the effective Lagrangian is related to the effective action as

$$\Gamma[A] = \int \mathrm{d}^4x\mathcal{L}_{\text{eff}}(F) = V\mathcal{L}_{\text{eff}}(F), \tag{16.4}$$

where V denotes the spacetime volume.

(**a**) At one-loop order, we can treat the electromagnetic field as a non-dynamical background, and integrate over the fermion field. Define the effective action $\Gamma[A]$ for the background field A as

$$Z = \int \mathcal{D}Ae^{i\Gamma[A]}. \tag{16.5}$$

Show that the resulting one-loop correction to the effective action reads

$$\Gamma^{(1)}[A] = -i\log\det(i\not{D} - m). \tag{16.6}$$

This is the fermion determinant, encoding all one-loop corrections from virtual electrons in the background field.

Hints:

- Gaussian path integrals are analogous to ordinary Gaussian integrals.
- Without gravity, constants in the effective action, even if they are infinite, do not contribute to the physics, and can be neglected. You can do this in every part of this sheet.

(b) As it is simpler to compute determinants of scalars, let's rewrite the determinant. Show that we can express the one-loop contribution to the effective action as

$$\Gamma^{(1)}[A] = -\frac{i}{2} \log\det(\not{D}^2 + m^2). \tag{16.7}$$

Hint: Use the fact that the operator $i\not{D} - m$ is hermitian. For Hermitian $\mathcal{O}$, it is known that $\log\det\mathcal{O} = \log[\det(\mathcal{O}^\dagger\mathcal{O})]/2$.

To simplify the problem, we use the proper-time representation of the effective action by expressing the Logarithm as

$$\log\mathcal{O} = -\int_0^\infty \frac{\mathrm{d}s}{s} e^{-s\mathcal{O}} + \text{const}. \tag{16.8}$$

(c) Demonstrate that the proper-time representation of Eq. (16.7) reads

$$\Gamma^{(1)}[A] = \frac{i}{2}\int_0^\infty \frac{\mathrm{d}s}{s} e^{-sm^2} \operatorname{tr}\left(e^{-s\not{D}^2}\right). \tag{16.9}$$

Computing the effective Lagrangian comes down to evaluating the trace of the operator $U = e^{-iHs}$, with the analogue of a Hamiltonian

$$H \equiv \not{D}^2. \tag{16.10}$$

This is why Eq. (16.8) is called proper-time representation: The operator D^2 is the generator of translations in $\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$. Therefore, the operator U is a translation in proper time and s, the Schwinger proper time, is the translation parameter.

(d) Show that

$$H = D^2 + \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu} \equiv H_{\text{kin}} + H_{\text{spin}}, \tag{16.11}$$

where $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$, and H_{kin} and H_{spin} denote the kinetic term and the coupling of spin to the electromagnetic field, respectively.

The purpose of this exercise is to compute the trace for $F_{\mu\nu}$ =const. Then, the spin-interaction Hamiltonian commutes with the kinetic term such that

$$\operatorname{tr}\left(e^{-sH}\right) = \operatorname{tr}\left(e^{-sH_{\text{kin}}}\right)\operatorname{tr}\left(e^{-sH_{\text{spin}}}\right). \tag{16.12}$$

(e) Show that

$$\operatorname{tr}\left(e^{-\frac{es}{2}F_{\mu\nu}\sigma^{\mu\nu}}\right) = 4\cos(esa)\cosh(esb), \tag{16.13}$$

where

$$a^2 = \sqrt{\mathcal{F}^2 + \mathcal{G}^2} - \mathcal{F}, \qquad b^2 = \sqrt{\mathcal{F}^2 + \mathcal{G}^2} + \mathcal{F}, \tag{16.14}$$

where we defined $\mathcal{G} \equiv F_{\mu\nu}\tilde{F}^{\mu\nu}/4$, and the dual field strength $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}/2$. Inasmuch as $a = 0$ if $\vec{E} = 0$ and $b = 0$ if $\vec{B} = 0$, we can understand a to largely measure electric contributions to the field strength, while b largely measures the magnetic ones.

Hints:

- H_{spin} is position independent, so what do we have to trace over?
- On the way, derive that $(F\sigma)^2 = 8(\mathcal{F} + i\gamma^5\mathcal{G})$. You can use the fact that

$$\{\sigma^{\mu\nu}, \sigma^{\rho\sigma}\} = 2\left(g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma} + i\gamma^5\epsilon^{\mu\nu\rho\sigma}\right), \tag{16.15}$$

  and that $(\gamma^5)^2 = 1$ and $\text{tr}\gamma^5 = 0$.
- The result of the trace has to be Lorentz invariant. Are there any Lorentz invariants which contain odd powers in $F_{\mu\nu}$?

Time for the last step. The trace of H_{kin} is best computed in Euclidean signature and afterwards analytically continued back. This amounts to the transformation $t \to -ix_0$, $\partial_t \to i\partial_0$. Thus, we transform

$$D^2 = \eta^{\mu\nu}D_\mu D_\nu \to -\delta_{AB}D_A D_B \equiv -(D_A)^2, \tag{16.16}$$

where indices A, B are four-dimensional Euclidean indices. Note that raising and lowering of indices is not required in Euclidean signature

(f) Show that the operator H_{kin} can be split into two commuting operators $H_{\text{kin},a}$ and $H_{\text{kin},b}$ by an orthogonal transformation such that

$$\text{tr}\left(e^{-sH_{\text{kin}}}\right) = \text{tr}\left(e^{-sH_{\text{kin},a}}\right)\text{tr}\left(e^{-sH_{\text{kin},b}}\right). \tag{16.17}$$

From here on, we will treat these two Hamiltonians jointly by using the shorthand notation $I = a, b$, to mean either of the two.

Hints:

- Antisymmetric matrices, like F_{AB}, can be put into Darboux-form, *i. e.* into non-mixing 2-by-2 antisymmetric blocks, by an orthogonal transformation.
- Use without proof that in four dimensions and in Euclidean signature, the matrix $F_{\mu\rho}F^{\rho\nu}$ has the eigenvalues $-a^2$ and $-b^2$.
- If F_{AB} =const, we can express the gauge field in a Landau-type gauge (show that!), where

$$A = ax_0\text{d}x_1 + bx_2\text{d}x_3. \tag{16.18}$$

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An operator trace formally amounts to a sum over all eigenvalues of an operator, including the multiplicity if the operator has degenerate eigenstates, namely

$$\text{tr}\left(e^{-sH_{\text{kin},I}}\right) = \sum_n M_{I,n}e^{-sE_{I,n}}, \tag{16.19}$$

where the $E_{I,n}$ are the eigenvalues of $H_{\text{kin},I}$, and $M_{I,n}$ is the multiplicity of eigenstate $|E_{I,n}\rangle$ (recall that $I = a, b$) and n can collectively stand for different quantum numbers. Note, though, that operators can have continuous spectra.

(g) Compute the eigenvalues of $H_{\text{kin},I}$.

Hint: You can reduce the problem to that of a one-dimensional quantum harmonic oscillator.

(**h**) There is something fishy going on with these eigenvalues. What is the multiplicity?

Don't despair! We have seen this kind of infinity before. Recall that we want to obtain the effective Lagrangian – not the effective action. Let us, for the moment, put our theory into a box. What we found is that the multiplicity scales with the side length of that box.
The number of allowed values of k provides the multiplicity but k also shifts the centre of motion of the harmonic oscillator. Put the two two-dimensional systems into quadratic boxes of side length L positioned such that the edges are at $(x_0, x_1) = (0, 0)$ and $(x_0, x_1) = (L_a, L_a)$ as well as $(x_2, x_3) = (0, 0)$ and $(x_2, x_3) = (L_b, L_b)$, with periodic boundary conditions. As a result, the whole theory is confined to a hypercube of box length L

(**i**) Estimate the number of states at fixed n, *i. e.* $M_{I,n}$. by requiring that the centre of motion for allowed k has to be inside the box. You should obtain

$$M_{I,n} = \frac{eIL^2}{2\pi}. \tag{16.20}$$

This appears to be sleight of hand, but is actually exact in the limit $L \to \infty$ that we will take in the end. Why? **Hint:** To answer the "why"-question, consider that $\bar{\psi}$ is the eigenfunction of the one-dimensional harmonic-oscillator Hamiltonian with shifted centre of motion.

(**j**) Show that

$$\operatorname{tr}\left(e^{-sH_{\text{kin}}}\right) = V_{\text{E}} \frac{e^2 ab}{(4\pi)^2 \sinh(esa) \sinh(esb)}, \tag{16.21}$$

where $V_{\text{E}} = L^4$ is the volume of the hypercube – the E here stands for Euclidean signature. Wick rotate back to Lorentzian signature, and provide the resulting effective Lagrangian

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{cl}} - \frac{e^2 ab}{8\pi^2} \int_0^\infty \frac{\mathrm{d}s}{s} e^{-sm^2} \cot(esa) \coth(esb), \tag{16.22}$$

where $\mathcal{L}_{\text{cl}} = -\mathcal{F}$.

Hint: Electric and magnetic fields behave differently under Wick rotation, namely $\vec{E}_{\text{E}} = i\vec{E}_{\text{L}}$ but $\vec{B}_{\text{E}} = \vec{B}_{\text{L}}$, where E stands for Euclidean and L for Lorentzian signature. How does the volume change under Wick rotation?

(**k**) The integral in Eq. (16.22) is divergent for $s \to 0$, *i. e.* in the UV. Renormalize it by subtracting solely the divergent part, *i. e.* do minimal subtraction. You should obtain

$$\mathcal{L}_{\text{eff,ren}} = \mathcal{L}_{\text{cl}} - \frac{1}{8\pi^2} \int_0^\infty \frac{\mathrm{d}s}{s^3} e^{-sm^2} \left(e^2 s^2 ab \cot(esa) \coth(esb) - 1 - \frac{e^2 s^2 (b^2 - a^2)}{3} \right). \tag{16.23}$$

(**l**) **Time for physics:** Before we tackle the full integral, let's do a simplification. Consider only magnetic fields so that $\mathcal{G} = 0$ and $\mathcal{F} > 0$. Expand in $\mathcal{F}$ inside the integral and compute the lowest-order contribution, namely

$$\mathcal{L}_{\text{eff}}^{(1)}|_{\mathcal{G}=0,\mathcal{F}>0} = \frac{2}{45\pi^2} \frac{e^4}{m^4} \mathcal{F}^2. \tag{16.24}$$

Why and in which regime can we expand inside the integral? Have a closer look at the resulting term: What kind of interaction did we get, *i. e.* what does backreaction do to the background fields? What would you expect at higher orders in the expansion? What does this mean for gravity?

The integral Eq. (16.23) covers the whole positive real line, where the integrand has an infinite number of poles. Thus it requires some work to be well-defined. Call the integrand

$$f(s) = \frac{e^{-sm^2}}{s^3}\left(e^2 s^2 ab \cot(esa)\coth(esb) - 1 - \frac{e^2 s^2 (b^2-a^2)}{3}\right). \tag{16.25}$$

We can render the integral well defined by shifting the poles into the complex plane, *i. e.* by considering $s \to s + i\epsilon$ and computing

$$\int_0^\infty \mathrm{d}s f(s+i\epsilon), \tag{16.26}$$

understood as a contour integral.

(m) Compute the imaginary part of the effective Lagrangian. You should obtain

$$\mathrm{Im}\mathcal{L}_{\text{eff}} = -\frac{e^2 ab}{8\pi}\sum_{j=1}^{\infty}\frac{e^{-\frac{j\pi m^2}{ae}}\coth\frac{j\pi b}{a}}{j\pi}. \tag{16.27}$$

Hint: The integrand is such that $\bar{f}(z) = f(\bar{z})$ for complex z, so the integral you have to evaluate is

$$\mathrm{Im}\int_0^\infty \mathrm{d}s f(s) = \frac{1}{2i}\int_0^\infty \mathrm{d}s\left(f(s+i\epsilon) - f(s-i\epsilon)\right). \tag{16.28}$$

This combined integral can be solved with the residue theorem.

Think before reading on: What could it mean that the imaginary part of the effective action is non-zero?

(n) Consider the time evolution of the vacuum. If we treat the electromagnetic field as a classical background, the amplitude describing vacuum being unchanged is given by the partition function

$$\langle 0_{\text{in}}|0_{\text{out}}\rangle = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS_{\text{QED}}}. \tag{16.29}$$

How does the probability of the vacuum staying the vacuum relate to the effective action? What does it mean if that probability is smaller than one? What does the quantity

$$\gamma = \frac{2\mathrm{Im}\Gamma[A]}{V} = 2\mathrm{Im}\mathcal{L}_{\text{eff}} \tag{16.30}$$

measure?

Hint: Revisit the motivation at the top of the sheet.

(a) Treating the gauge field as an external field, we can write

$$Z = \int \mathcal{D}A e^{iS_{\text{cl}}}\left(\int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{i\int_x \bar{\psi}(i\not{D}-m)\psi}\right), \tag{16.31}$$

with the classical (*i. e.* tree-level) action $S_{\text{cl}} = -\int_x F_{\mu\nu}F^{\mu\nu}/4$. The involved integral is Gaussian and can be solved explicitly, yielding

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{i\int_x \bar{\psi}(i\not{D}-m)\psi} = \mathcal{N}\det(i\not{D}-m), \tag{16.32}$$

where we introduced some (infinite) normalization constant $\mathcal{N}$. Thus, the effective action satisfies

$$e^{i\Gamma[A]} = e^{iS_{\rm cl}}\mathcal{N}\det(i\not{D} - m). \tag{16.33}$$

Thus, we obtain for the effective action

$$\Gamma[A] = S_{\rm cl} - i\log\det(i\not{D} - m) - i\log\mathcal{N}. \tag{16.34}$$

Neglecting the infinite constant $\mathcal{N}$ – constants, even if they are infinite, are irrelevant to the effective action in the absence of gravity –, we obtain the one-loop correction

$$\Gamma^{(1)}[A] = \Gamma[A] - S_{\rm cl} = -i\log\det(i\not{D} - m). \tag{16.35}$$

(**b**) As the operator $i\not{D} - m$ is hermitian, we can re-express the logarithm of its determinant as the sum of itself and its hermitian conjugate, namely

$$\log\det(i\not{D} - m) = \frac{1}{2}\left[\log\det(i\not{D} - m) + \log\det(-i\not{D} - m)\right], \tag{16.36}$$

$$= \frac{\log[\det(i\not{D} - m)\det(-i\not{D} - m)]}{2}, \tag{16.37}$$

$$= \frac{\log\det[(i\not{D} - m)(-i\not{D} - m)]}{2}, \tag{16.38}$$

$$= \frac{\log\det(\not{D}^2 + m^2)}{2}. \tag{16.39}$$

Thus, we obtain

$$\Gamma^{(1)}[A] = -\frac{i}{2}\log\det(\not{D}^2 + m^2). \tag{16.40}$$

(**c**) Using the fact that for any operator $\log\det\mathcal{O} = \operatorname{tr}\log\mathcal{O}$, we can rewrite

$$\Gamma^{(1)}[A] = -\frac{i}{2}\operatorname{tr}\log(\not{D}^2 + m^2), \tag{16.41}$$

$$= \frac{i}{2}\operatorname{tr}\int_0^\infty \frac{\mathrm{d}s}{s} e^{-s(\not{D}^2 + m^2)} + \text{const}, \tag{16.42}$$

$$= \frac{i}{2}\int_0^\infty \frac{\mathrm{d}s}{s} e^{-sm^2}\operatorname{tr}\left(e^{-s\not{D}^2}\right) + \text{const}. \tag{16.43}$$

(**d**) We compute directly

$$H_{\rm spin} = \gamma^\mu\gamma^\nu D_\mu D_\nu, \tag{16.44}$$

$$= (\eta^{\mu\nu} - i\sigma^{\mu\nu})D_\mu D_\nu, \tag{16.45}$$

$$= D^2 - i\sigma^{\mu\nu}D_{[\mu}D_{\nu]}, \tag{16.46}$$

$$= D^2 - \frac{i}{2}\sigma^{\mu\nu}[D_\mu, D_\nu], \tag{16.47}$$

$$= D^2 + \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}, \tag{16.48}$$

where we used the anticommutation relations of the γ-matrices, that $\sigma^{\mu\nu}$ is antisymmetric in its indices, and that $F_{\mu\nu} \equiv [D_\mu, D_\nu]/ie$ by definition.

(**e**) The trace that we have to compute is solely in spinor space. First, we expand

$$\operatorname{tr}\left(e^{-\frac{es}{2}F_{\mu\nu}\sigma^{\mu\nu}}\right) = \sum_{n=0}^\infty \frac{\left(\frac{-es}{2}\right)^n}{n!}\operatorname{tr}\left[(F\sigma)^n\right]. \tag{16.49}$$

As the resulting expression is local and Lorentz invariant, it cannot depend on odd powers of the field strength – a Lorentz invariant, local operator of odd power in field strength does not exist. Therefore, $\mathrm{tr}[(F\sigma)^{2n+1}] = 0$ for integer n. As a result, we can write

$$\mathrm{tr}\left(e^{-\frac{es}{2}F_{\mu\nu}\sigma^{\mu\nu}}\right) = \sum_{n=0}^{\infty} \frac{\left(\frac{-es}{2}\right)^{2n}}{(2n)!} \mathrm{tr}\left[(F\sigma)^{2n}\right]. \tag{16.50}$$

Next, let's compute $(F\sigma)^2$, which we can express as

$$(F\sigma)^2 = F_{\mu\nu}F_{\rho\sigma}\sigma^{\mu\nu}\sigma^{\rho\sigma}, \tag{16.51}$$
$$= \frac{1}{2}F_{\mu\nu}F_{\rho\sigma}\{\sigma^{\mu\nu}, \sigma^{\rho\sigma}\}, \tag{16.52}$$
$$= 2F_{\mu\nu}F^{\mu\nu} + i\gamma^5\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}, \tag{16.53}$$
$$= 8\left(\mathcal{F} + i\gamma^5\mathcal{G}\right). \tag{16.54}$$

Thus, the kinds of traces, we have to take are

$$\mathrm{tr}[(F\sigma)^{2n}] = 8^n \mathrm{tr}[\left(\mathcal{F} + i\gamma^5\mathcal{G}\right)^n], \tag{16.55}$$
$$= 8^n \sum_{m=0}^{n} \binom{n}{m} \mathrm{tr}\left[\mathcal{F}^{n-m}(i\gamma^5\mathcal{G})^m\right]. \tag{16.56}$$

Now we use the hints on γ^5 to arrive at

$$\mathrm{tr}[(F\sigma)^{2n}] = 8^n \sum_{m=0}^{n/2} \binom{n}{2m} \left[\mathcal{F}^{n-2m}(i\mathcal{G})^{2m}\right] \mathrm{tr}[\mathbb{1}_{4\times4}]. \tag{16.57}$$

This is a binomial sum where all the odd powers of $i\mathcal{G}$ have been projected out. Thus, we obtain

$$\mathrm{tr}[(F\sigma)^{2n}] = \frac{\mathrm{tr}[\mathbb{1}_{4\times4}]}{2} 8^n \left[(\mathcal{F} + i\mathcal{G})^n + (\mathcal{F} - i\mathcal{G})^n\right]. \tag{16.58}$$

Thus, with $\mathrm{tr}[\mathbb{1}_{4\times4}] = 4$ the whole sum becomes

$$\mathrm{tr}\left(e^{-\frac{es}{2}F_{\mu\nu}\sigma^{\mu\nu}}\right) = 2\sum_{n=0}^{\infty} \frac{\left(-\frac{\sqrt{8}es}{2}\right)^{2n}}{(2n)!} \left[(\mathcal{F} + i\mathcal{G})^n + (\mathcal{F} - i\mathcal{G})^n\right], \tag{16.59}$$
$$= 2\left[\cosh\left(\frac{es}{2}\sqrt{8(\mathcal{F} + i\mathcal{G})}\right) + \cosh\left(\frac{es}{2}\sqrt{8(\mathcal{F} - i\mathcal{G})}\right)\right]. \tag{16.60}$$

Using the definitions of a and b, we obtain

$$\sqrt{\mathcal{F} \pm i\mathcal{G}} = \frac{1}{\sqrt{2}}(ia \mp b), \tag{16.61}$$

such that

$$\mathrm{tr}\left(e^{-\frac{es}{2}F_{\mu\nu}\sigma^{\mu\nu}}\right) = 2\left[\cos\left(es(a + ib)\right) + \cos\left(es(a - ib)\right)\right], \tag{16.62}$$
$$= 4\cos(esa)\cosh(esb), \tag{16.63}$$

where we used a trigonometric addition formula in the last equality.

(**f**) F_{AB} is an antisymmetric matrix also in Euclidean signature. Any antisymmetric $2n$-by-$2n$ matrix can be brought into block diagonal form by an orthogonal transformation, where the n blocks are themselves 2-by-2 matrices. This block-diagonalization amounts to diagonalizing the square of the antisymmetric matrix. Then, nontrivial entries of the resulting 2-by-2 blocks are the eigenvalues of that square of the antisymmetric matrix.

As the background is four-dimensional Euclidean space, *i. e.* symmetric under $O(4)$-transformations aka $4D$ rotations, we are allowed to compute eigenvalues in such a frame. Then, the field strength splits the tensor into electric and magnetic fields. The 2-by-2 blocks are characterized by the eigenvalues of $F_{AB}F_{BC}$, namely $-a^2$ and $-b^2$ (see hint). Consequently, we can express the field strength as

$$F_{AB} = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix} . \tag{16.64}$$

Indeed, if we choose the Landau-type gauge given in Eq. (16.18), we obtain this form of the field strength from

$$F_{AB} = \partial_A A_B - \partial_B A_A. \tag{16.65}$$

Thus, after the orthogonal transformation and in Landau-type gauge, the operator H_{kin} reads

$$H_{\text{kin}} = H_{\text{kin},a} + H_{\text{kin},b}, \tag{16.66}$$

where

$$H_{\text{kin},a} = (-i\partial_0)^2 + (-i\partial_1 + eax_0)^2 \,, \tag{16.67}$$

$$H_{\text{kin},b} = (-i\partial_2)^2 + (-i\partial_3 + ebx_2)^2 \,. \tag{16.68}$$

As these two Hamiltonians commute with each other, we can trace over them independently and split their appearance in the exponent such that

$$\text{tr}\left(e^{-sH_{\text{kin}}}\right) = \text{tr}\left(e^{-s(H_{\text{kin},A}+H_{\text{kin},B})}\right) = \text{tr}\left(e^{-sH_{\text{kin},A}}\right)\text{tr}\left(e^{-sH_{\text{kin},B}}\right). \tag{16.69}$$

(**g**) Each $H_{\text{kin,I}}$ (hereafter $I = a, b$) describes a Landau-level system in Landau gauge (*i. e.* charged particle in a constant background magnetic field). Those depend on x_1 and x_3 only through derivatives. Thus, we can express the eigenfunctions of the Hamiltonians as

$$\psi_a(x_0, x_1) = e^{ik_a x_1}\bar{\psi}_a(x_0), \qquad \psi_b(x_2, x_3) = e^{ik_b x_3}\bar{\psi}_b(x_2), \tag{16.70}$$

with the continuous quantum numbers k_I, and where the ψ_I are defined such that $H_I\psi_I = E_I\psi_I$. As a result, the Hamiltonians become

$$H_{\text{kin},a}\bar{\psi}_a = \left[(-i\partial_0)^2 + (k_a + eax_0)^2\right]\bar{\psi}_a, \tag{16.71}$$

$$H_{\text{kin},b}\bar{\psi}_b = \left[(-i\partial_2)^2 + (k_b + ebx_2)^2\right]\bar{\psi}_b. \tag{16.72}$$

These two Hamiltonians describe harmonic oscillators whose centre of motion is shifted with respect to the origin by an amount which depends on the quantum number k. This shift cannot be translated away, because we cannot make sense of coordinates which depend on the state of a system, namely the value of k. For the spectrum of the Hamiltonian, the position of the centre of motion is irrelevant, though. To get the spectrum in the correct units, let us define dimensionless coordinates $y_a = \sqrt{ea}x_0$ and $y_b = \sqrt{eb}x_2$ to bring the Hamiltonians into canonical form (recall $I = a, b$)

$$H_{\text{kin},I}\bar{\psi}_I = eI\left[(-i\partial_{y_I})^2 + (y_I - y_{I,c})^2\right]\bar{\psi}_I, \tag{16.73}$$

where $y_{I,\mathrm{c}} = -k_I/\sqrt{eI}$. As a result, we can read off the eigenvalues as the eigenvalues of a one-dimensional harmonic oscillator, namely

$$E_{\mathrm{kin},I,n_I} = 2eI\left(n_I + \frac{1}{2}\right). \tag{16.74}$$

(**h**) The problem is two-dimensional, but there is only one quantum number in the eigenvalues. The other quantum number, k, only appears in the position of the centre of motion. Thus, the multiplicity is infinite.

(**i**) In a box of length L the eigenvalues of the operators $-i\partial_1$ and $-i\partial_3$ are quantized as

$$k_I = \frac{2\pi m_I}{L}, \tag{16.75}$$

where m_I is a new discrete quantum number,*i. e.* a (possibly negative) integer. That's just the free-particle-in-a-box problem we all know and love/hate. The dimensionful position of the centre of motion is

$$x_{I,\mathrm{c}} = \frac{y_{I,\mathrm{c}}}{\sqrt{eI}} = -\frac{k_I}{eI} = -\frac{2\pi m_I}{eIL}. \tag{16.76}$$

To not shift the centre out of the box, m_I has to satisfy

$$0 \leq x_{I,\mathrm{c}} = -\frac{2\pi m_I}{eIL} \leq L. \tag{16.77}$$

Thus, we have

$$-\frac{eIL^2}{2\pi} \leq m_I \leq 0. \tag{16.78}$$

As m_I is an integer, we can have

$$M_{I,n} = \frac{eIL^2}{2\pi} + \mathcal{O}(1) \tag{16.79}$$

states in the box in any level n. In the limit $L \to \infty$, we can neglect the order-one contribution. Thus, in the box we obtain the multiplicity

$$M_{I,n} = \frac{eIL^2}{2\pi}, \tag{16.80}$$

which is independent of n.

Why is this an accurate state counting when $L \to \infty$? Harmonic-oscillator eigenstates always contain a Gaussian

$$\bar{\psi}_I \propto e^{-\frac{(y_I - y_{I,\mathrm{c}})^2}{2}} = e^{-\frac{(x_{1,3} - x_{I,\mathrm{c}})^2}{2eI}}, \tag{16.81}$$

where here the notation $x_{1,3}$ means that $x_{1,3} = x_1$ if $I = a$ and x_3 if $I = b$. Thus, the standard deviation of the states in comparison to the size of the box gets smaller and smaller with increasing L as

$$\frac{\Delta x_{1,3}}{L} = \frac{\sqrt{eI}}{L}, \tag{16.82}$$

i. e. they are extremely sharply peaked for large L. We can safely count a state as allowed if the boundary loss is negligible, *i. e.* if

$$\int_{x_{1,3} \notin [0,L]} |\bar{\psi}|^2 \leq \epsilon \tag{16.83}$$

for some small fixed number ϵ. We can safely count them as not allowed if they are positioned almost exclusively outside of the box, *i. e.* if

$$\int_{x_{1,3}\notin[0,L]} |\bar{\psi}|^2 \geq 1-\epsilon. \tag{16.84}$$

The only states that we cannot be sure about are those which satisfy neither of these two inequalities. But as the states are sharply peaked for large L (*i. e.* their standard deviation is independent of L), however small the value of ϵ is one chooses to work with, the number of states we are not sure about always scales as $\mathcal{O}(1)$. As we already discussed above, the multiplicity given in Eq. (16.79) goes like $M_{I,n} \propto L^2$. Thus, order-one corrections are negligible in the limit $L \to \infty$, and the number density we computed is exact.

(**j**) We have to sum the eigenvalues, *i. e.*

$$\begin{aligned}
\operatorname{tr}\left(e^{-sH_{\mathrm{kin},I}}\right) =& \sum_{n=0}^{\infty} M_{I,n} e^{-sE_{n,I}}, & (16.85)\\
=& \frac{eIL^2}{2\pi} \sum_{n=0}^{\infty} e^{-2esI\left(n+\frac{1}{2}\right)}, & (16.86)\\
=& \frac{eIL^2}{2\pi} e^{-esI} \sum_{n=0}^{\infty} e^{-2esIn}, & (16.87)\\
=& \frac{eIL^2}{2\pi} \frac{e^{esI}}{e^{2esI}-1}, & (16.88)\\
=& \frac{eIL^2}{4\pi \sinh(esI)}. & (16.89)
\end{aligned}$$

Thus, in total we obtain

$$\begin{aligned}
\operatorname{tr}\left(e^{-sH_{\mathrm{kin}}}\right) =& \operatorname{tr}\left(e^{-sH_{\mathrm{kin},a}}\right) \operatorname{tr}\left(e^{-sH_{\mathrm{kin},b}}\right), & (16.90)\\
=& \frac{e^2abL^4}{(4\pi)^2 \sinh(esa)\sinh(esb)} & (16.91)\\
=& V_{\mathrm{E}} \frac{e^2ab}{(4\pi)^2 \sinh(esa)\sinh(esb)}. & (16.92)
\end{aligned}$$

As the hint together with Eq. (16.64) indicates, Wick rotation shifts $a \to ia$ and $b \to b$. The volume has to be understood as a volume integral

$$V_{\mathrm{E}} = \int \mathrm{d}x_0 \mathrm{d}\vec{x} = i \int \mathrm{d}t \mathrm{d}\vec{x} = iV. \tag{16.93}$$

The effective Lagrangian then reads

$$\begin{aligned}
\mathcal{L}_{\mathrm{eff}} =& \lim_{V\to\infty}\left(S_{\mathrm{cl}} + \frac{\Gamma^{(1)}[A]}{V}\right), & (16.94)\\
=& \mathcal{L}_{\mathrm{cl}} + \frac{i}{2V}\int_0^\infty \frac{\mathrm{d}s}{s} \operatorname{tr}\left(e^{-\frac{es}{2}F_{\mu\nu}\sigma^{\mu\nu}}\right) \operatorname{tr}\left(e^{-sH_{\mathrm{kin}}}\right), & (16.95)\\
=& \mathcal{L}_{\mathrm{cl}} - \frac{e^2ab}{8\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s} e^{-sm^2} \cot(esa)\coth(esb). & (16.96)
\end{aligned}$$

(**k**) For small s, the divergent contributions to the integrand of Eq. (16.22) are

$$\frac{e^{-sm^2}}{s}\cot(esa)\coth(esb) \simeq \frac{1}{e^2ab}e^{-sm^2}\left[\frac{1}{s^3}+\frac{e^2(b^2-a^2)}{3s}\right] + \text{finite terms.} \tag{16.97}$$

Thus, we can remove the divergences by subtracting off the divergent terms such that

$$\mathcal{L}_{\text{eff,ren}} = \mathcal{L}_{\text{cl}} - \frac{1}{8\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s^3}e^{-sm^2}\left(e^2s^2ab\cot(esa)\coth(esb) - 1 - \frac{e^2s^2(b^2-a^2)}{3}\right). \tag{16.98}$$

(**l**) If $\mathcal{G}=0$ and $\mathcal{F}>0$, we immediately obtain $a=0$, $b=2\mathcal{F}$. Thus, the effective Lagrangian becomes

$$\mathcal{L}_{\text{eff,ren}}|_{\mathcal{G}=0,\mathcal{F}>0} = \mathcal{L}_{\text{cl}} - \frac{1}{8\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s^3}e^{-sm^2}\left(e^2sb\coth(2es\sqrt{\mathcal{F}}) - 1 - \frac{4e^2s^2\sqrt{\mathcal{F}}}{3}\right). \tag{16.99}$$

Expanding in $\mathcal{F}$ to lowest order, we obtain

$$\mathcal{L}^{(1)}_{\text{eff}}|_{\mathcal{G}=0,\mathcal{F}>0} = \frac{2}{45\pi^2}e^4\mathcal{F}^2\int_0^\infty e^{-m^2s}s\mathrm{d}s, \tag{16.100}$$

$$= \frac{2}{45\pi^2}\frac{e^4\mathcal{F}^2}{m^4}. \tag{16.101}$$

Why and in which regime were we allowed to expand inside the integral? Besides the dependence of the field strength, the integrand contains a factor e^{-sm^2}, which suppresses large-s contributions to the integral. The expansion is fine, as long as the part of the domain where it does not converge is within this suppressed regime. Thus, the expansion is fine as long as

$$\frac{e^2\mathcal{F}}{m^2} \ll 1. \tag{16.102}$$

What kind of interaction did we get? Recall that $-\mathcal{F}$ is the classical QED Lagrangian. The new interaction term, then, contains four photons. Thus, it is a photon-photon interaction, which is suppressed by the mass of the electron and the dimensionless strength of the photon-electron coupling (e). This is exactly what one would expect from a theory where we integrated out a particle that mediates interactions. In this case, we can see that the photon-photon interaction comes from shrinking down a box diagram with internal fermion loop to a point.

What would we expect at higher order in the expansion? The expansion is in $e\mathcal{F}/m^2$, so we expect higher- and higher-order contributions introducing many-photon self interactions.

What does this mean for gravity? The analogous effect should also apply to gravity, where integrating out other fields, we obtain higher-curvature corrections to the gravitational effective action, which are suppressed by powers of $\sqrt{g}/m$, where $g=\mu^2G$ is the dimensionless Newton coupling and μ is a characteristic energy scale, at which we probe the theory. We have to make this detour because G, in contrast to e, is dimensionful. Thus, if we probe at energies smaller than m (which we have to to expand as we did above), we can say that we have a suppression $\leq \sqrt{G}m = m/m_{\text{Pl}}$. The first kind of contribution to the effective, we expect to arise from backreaction of particles of mass m on the geometry is of the form

$$\left(\frac{m}{m_{\text{Pl}}}\right)^2\left(\alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}\right), \tag{16.103}$$

and all of this without quantum gravity being involved!

(**m**) In Eq. (16.28), we subtract the integral with the poles shifted above the real axis from one with the poles shifted below the real axis. This expression can be interpreted as a contour integral where the contour is a curve going from $s = i\epsilon$ to $s = \infty + i\epsilon$, down to $s = \infty - i\epsilon$, then back to $s = -i\epsilon$, and up to $s = i\epsilon$. In short, the contour is a rectangle in the complex plane encircling the poles on the real axis. This works because the integrand goes to zero at infinity, and so the integral from $s = \infty + i\epsilon$ to $s = \infty - i\epsilon$ vanishes in the limit $\epsilon \to 0$, as does the integral from $-i\epsilon$ to ϵ. Thus, the resulting contour encircles all poles of f on the real line, such that the imaginary part of the effective action is given by a sum over the residues of all poles

$$\mathrm{Im}\mathcal{L}_{\text{eff}} = -\frac{1}{8\pi} \sum_j \mathrm{Res}(f, s_j), \tag{16.104}$$

where s_j are the poles of $f(s)$. The cotangent has poles wherever its argument equals an integer multiple of π, so $s_j = j\pi/ea$. Thus, we obtain the result

$$\mathrm{Im}\mathcal{L}_{\text{eff}} = -\frac{e^2 ab}{8\pi} \sum_{j=1}^{\infty} \frac{e^{-\frac{j\pi m^2}{ae}} \coth \frac{j\pi b}{a}}{j\pi}. \tag{16.105}$$

As the hyperbolic cotangent asymptotes to one for large arguments, single elements of the sum decay exponentially for large j. Thus, the sum is convergent.

(**n**) Using Eqs. (16.1) and (16.5), we have

$$Z = \int \mathcal{D}A \langle 0_{\text{in}} | 0_{\text{out}} \rangle = \int \mathcal{D}A e^{i\Gamma[A]}. \tag{16.106}$$

Thus, the partition function describing the vacuum-vacuum amplitude reads

$$\langle 0_{\text{in}} | 0_{\text{out}} \rangle = e^{i\Gamma[A]} = e^{i\mathrm{Re}\Gamma[A]} e^{-\mathrm{Im}\Gamma[A]}. \tag{16.107}$$

If we now compute the probability of the vacuum remaining the vacuum, we obtain

$$|\langle 0_{\text{in}} | 0_{\text{out}} \rangle|^2 = e^{-2\mathrm{Im}\Gamma[A]} = e^{-2V\mathrm{Im}\mathcal{L}_{\text{eff}}} = e^{-\gamma V}. \tag{16.108}$$

Thus, the vacuum changes into a state which is not the vacuum with a probability $1 - e^{-\gamma V}$. It decays, and γ is the decay rate per unit volume. The relevant energy scale at which this happens is $\sim m$. Considering Eq. (16.27), we find that if the background electric field is strong enough such that $ea/m > 1$, the probability of decay is non-negligible. This is the famous Schwinger particle creation in strong electric backgrounds. Sure enough, this is an enormous electric field, but it is a prediction of QED. Does this remind you of something? Right, particles can also be created by gravitational fields. Indeed, the kind of Bogolyubov-coefficient computations we have done in the past can also be done for the Schwinger effect, exactly as they can be for gravitational backgrounds.