Quantum field theory in curved spacetime

Assignment 8 – June 23

Exercise 17: Heat kernels on curved backgrounds

Motivation: Last week, we encountered the trace of a differential operator. In curved backgrounds, such a trace is best calculated with so-called heat-kernel techniques. Here we get to know the heat kernels for a real scalar field.

This exercise is similar to last week's. However, Wick rotation in curved backgrounds is difficult (if not impossible). Therefore, we will stay Euclidean throughout this time. The partition function for a massive scalar on a curved Euclidean background reads

$$Z = \int \mathcal{D}\phi e^{-S_\phi}, \tag{17.1}$$

with the action

$$S_\phi = \frac{1}{2}\int \mathrm{d}^4x\sqrt{g}\left[g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + m^2\phi^2\right]. \tag{17.2}$$

The Euclidean effective action $\Gamma[g]$ is defined such that

$$e^{-\Gamma} = \int \mathcal{D}g e^{-S_{\mathrm{EH}}} Z, \tag{17.3}$$

with the Euclidean Einstein-Hilbert action

$$S_{\mathrm{EH}} = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{g}\left(-R+2\Lambda\right). \tag{17.4}$$

(a) As a throwback to last week's exercise, show that for constant metric the scalar contributes to the gravitational effective action as

$$\Gamma^{(1)}[g] = \frac{1}{2}\mathrm{tr}\log(-\Delta + m^2) + \mathrm{const.} \tag{17.5}$$

The constant part has to be renormalized and contributes to the cosmological constant, but we will ignore those contributions here. Again, we use the proper-time representation aka the Laplace transform of the logarithm, *c. f.* Eq. (16.8) yielding

$$\Gamma^{(1)}[g] = -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}s}{s} e^{-sm^2}\mathrm{tr}\left[e^{s\Delta}\right]. \tag{17.6}$$

We will now compute the trace with heat kernels. The heat kernel K_M of an operator M satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}s}K_M = -MK_M, \tag{17.7}$$

with the initial condition $K_M(0) = \mathbb{1}$. The trace of the heat kernel is the trace we want to compute. In position space, where $\mathbf{x} = (x_0, x_1, x_2, x_3)$, we can define $K(\mathbf{x}, \mathbf{x}', s) = \langle \mathbf{x}|K_M|\mathbf{x}'\rangle$, and represent Eq. (17.7) as

$$\frac{\mathrm{d}}{\mathrm{d}s}K(\mathbf{x}, \mathbf{x}', s) = -MK(\mathbf{x}, \mathbf{x}', s), \tag{17.8}$$

with the initial condition $K(\mathbf{x}, \mathbf{x}', 0) = \delta^{(4)}(\mathbf{x} - \mathbf{x}')$.

(**b**) Demonstrate that if we choose $M = -\Delta$, *i. e.* if

$$\frac{\mathrm{d}}{\mathrm{d}s} K(\mathbf{x}, \mathbf{x}', s) = \Delta_{\mathbf{x}} K(\mathbf{x}, \mathbf{x}', s), \tag{17.9}$$

we can write

$$\mathrm{tr} e^{s\Delta} = \int \mathrm{d}^4 x K(\mathbf{x}, \mathbf{x}, s). \tag{17.10}$$

Here, $\Delta_{\mathbf{x}}$ acts on the label $\mathbf{x}$ only. From here on, we don't use the subscript $\mathbf{x}$ on differential operators any more. All derivatives act on $\mathbf{x}$ unless otherwise stated.

(**c**) Let's start in flat space. Show that in a flat Euclidean background, Eq. (17.9) has the solution

$$K(\mathbf{x}, \mathbf{x}', s) = \frac{1}{(4\pi s)^2} \exp\left[-\frac{(\mathbf{x} - \mathbf{x}')^2}{4s} \right]. \tag{17.11}$$

Hint: In flat space, one can use Fourier methods.

The heat kernel is a function of the distance only. This makes sense because the result should be a scalar. Similarly, in curved space, the result can only depend on scalars constructed from (derivatives of) the squared geodesic distance and the curvature tensor. We work with the world function $\sigma(x, x')$ which equals one half of the squared geodesic distance and satisfies the differential equation

$$\frac{1}{2} \nabla_\mu \sigma(\mathbf{x}, \mathbf{x}') \nabla^\mu \sigma(\mathbf{x}, \mathbf{x}') = \sigma(\mathbf{x}, \mathbf{x}'). \tag{17.12}$$

We know that we can use normal coordinates in a finite neighbourhood of a point. Riemann normal coordinates, for example, yield a flat metric at that point plus corrections. Here, we do something similar. We make an ansatz

$$K(\mathbf{x}, \mathbf{x}', s) = \frac{1}{(4\pi s)^2} e^{-\frac{\sigma(\mathbf{x}, \mathbf{x}')}{2s}} \sum_{n=0}^{\infty} A_n(\mathbf{x}, \mathbf{x}') s^n, \tag{17.13}$$

where the coefficients $A_n(\mathbf{x}, \mathbf{x}')$ encapsulate the deviation from flat space.

(**d**) What does the boundary condition translate to in terms of the coefficients $A_n(\mathbf{x}, \mathbf{x}')$?

Using Eq. (17.12), one can show that the coefficients satisfy the recursive differential equation

$$\left[n - 2 + \frac{\Delta \sigma(\mathbf{x}, \mathbf{x}')}{2} \right] A_n(\mathbf{x}, \mathbf{x}') + \nabla^\mu \sigma(\mathbf{x}, \mathbf{x}') \nabla_\mu A_n(\mathbf{x}, \mathbf{x}') - \Delta A_{n-1}(\mathbf{x}, \mathbf{x}') = 0, \tag{17.14}$$

where $A_n = 0$ for $n < 0$.
In the end, we are not interested in $K(\mathbf{x}, \mathbf{x}', s)$ because the trace of the heat kernel

$$\mathrm{tr} K_M = \int \mathrm{d}^4 \mathbf{x} \sqrt{g} K(\mathbf{x}, \mathbf{x}, s) \tag{17.15}$$

contains only the diagonal elements of $K(\mathbf{x}, \mathbf{x}', s)$. Therefore, we do not need the coefficients $A_n(\mathbf{x}, \mathbf{x}')$, but only their coincidence limits

$$a_n(\mathbf{x}) = \lim_{\mathbf{x}' \to \mathbf{x}} A_n(\mathbf{x}, \mathbf{x}'). \tag{17.16}$$

Let's compute $a_1(x)$.

(**e**) Demonstrate that a_1 satisfies a differential equation whose solution requires the knowledge of $\lim_{\mathbf{x}'\to\mathbf{x}} \Delta A_0$.

Hint: Note that coincident limits and covariant derivatives do not commute. From here on, you can use without proof (or prove if you want) that Eq. (17.12) together with $\lim_{x'\to x} \sigma(x,x') = 0$ implies

$$\lim_{x'\to x} \nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_n}\sigma(x,x') = 0 \text{ for odd } n, \tag{17.17}$$

$$\lim_{x'\to x} \nabla_\mu\nabla_\nu\sigma(x,x') = g_{\mu\nu}, \tag{17.18}$$

$$\lim_{x'\to x} \nabla_\mu\nabla_\nu\nabla_\rho\nabla_\sigma\sigma(x,x') = -\frac{1}{3}\left(R_{\rho\mu\sigma\nu} + R_{\rho\nu\sigma\mu}\right). \tag{17.19}$$

(**f**) By acting with Δ on the differential equation for A_0 and taking the coincident limit, show that

$$\lim_{\mathbf{x}'\to\mathbf{x}} \Delta A_0 = \frac{R}{6}. \tag{17.20}$$

Show that as a result

$$a_1 = \frac{R}{6}. \tag{17.21}$$

(**g**) Going back to Eq. (17.6) show that the terms containing a_0 and a_1 are divergent. They contribute to the renormalization of constants of the background theory. Which are those?

At higher order in s, one obtains[a]

$$a_2 = \frac{R^2}{120} + \frac{R_{\mu\nu}R^{\mu\nu}}{60}. \tag{17.22}$$

(**h**) Verify that the resulting contribution to the renormalized effective action reads

$$\Gamma^{(1)}_{\text{ren}}[g] = \frac{\gamma_{\text{E}} + 2\log m}{120(4\pi)^2}\int \mathrm{d}^4x\sqrt{g}\left(\frac{R^2}{2} + R_{\mu\nu}R^{\mu\nu}\right), \tag{17.23}$$

with the Euler-Mascheroni constant γ_{E}. Indeed, we obtain higher-order corrections to general relativity from integrating out matter particles. What happens in the limit $m \to 0$?

Hint: Use a proper time cut-off, *i. e.* integrate over s from ϵ to infinity, and isolate the part which is finite in the limit $\epsilon \to 0$.

[a] The expression omits the $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ term by using the Gauss-Bonnet identity.

(**a**) We can rewrite the action as

$$S_\phi = \frac{1}{2}\int \mathrm{d}^4x\sqrt{g}\phi\left[-\Delta + m^2\right]\phi. \tag{17.24}$$

The resulting path integral is Gaussian, and can be evaluated immediately yielding

$$\Gamma^{(1)} = -\log\mathcal{N} - \log\left[\det\left(-\Delta + m^2\right)\right]^{-\frac{1}{2}}, \tag{17.25}$$

$$= \frac{1}{2}\log\det\left(-\Delta + m^2\right) + \text{const.} \tag{17.26}$$

(**b**) The differential equation defining the heat kernel, namely Eq. (17.7), together with the initial condition $K(0) = \mathbb{1}$ is solved by the operator

$$K_M = e^{-sM}. \tag{17.27}$$

The trace of a differential operator without further degrees of freedom (like e. g. spin) can be taken in position space, *i. e.*

$$\mathrm{tr} e^{s\Delta} = \int \mathrm{d}^4x \langle x|e^{s\Delta}|x\rangle, \tag{17.28}$$

$$= \int \mathrm{d}^4x \langle x|K_{-\Delta}|x\rangle. \tag{17.29}$$

(**c**) The initial condition is satisfied by definition of the delta function. In Fourier space, we can write Eq. (17.9) as

$$\frac{\mathrm{d}}{\mathrm{d}s}\tilde{K}(\mathbf{k}, \mathbf{x}', s) = -k^2\tilde{K}(\mathbf{k}, \mathbf{x}', s). \tag{17.30}$$

This equation is solved by

$$\tilde{K}(\mathbf{k}, \mathbf{x}', s) = f(\mathbf{x}', \mathbf{k})e^{-k^2 s}, \tag{17.31}$$

for any regular function f. The function f is determined by the initial condition, namely $K(\mathbf{k}, \mathbf{x}', 0) = e^{-i\mathbf{k}\mathbf{x}'}$. Going back to position space, we obtain

$$K(\mathbf{x}, \mathbf{x}', s) = \int \frac{\mathrm{d}^4\mathbf{k}}{(2\pi)^4} e^{-\mathbf{k}^2 s + i\mathbf{k}(\mathbf{x}-\mathbf{x}')} = \frac{1}{(4\pi s)^2} e^{-\frac{(\mathbf{x}-\mathbf{x}')^2}{4s}}. \tag{17.32}$$

(**d**) One definition of the delta function in four-dimensional flat space is

$$\delta(\mathbf{x}) = \lim_{\epsilon\to 0} \frac{e^{-\frac{\mathbf{x}^2}{2\epsilon}}}{(2\pi\epsilon)^2}. \tag{17.33}$$

If we choose $\epsilon = 2s$, we exactly obtain the limit

$$K(\mathbf{x}, \mathbf{x}', 0) = \lim_{\epsilon\to 0} \frac{1}{(2\pi\epsilon)^2} e^{-\frac{\sigma}{2\epsilon}} a_0. \tag{17.34}$$

This is a four-dimensional Δ-function with the Euclidean distance function replaced by the general Riemannian one. However, the result is only non-vanishing if the Riemannian distance vanishes, which is a limit in which the Riemannian nature is irrelevant (locally Riemannian geometry is Euclidean). Thus, we obtain

$$K(\mathbf{x}, \mathbf{x}', 0) = \delta^{(4)}(\mathbf{x} - \mathbf{x}')a_0. \tag{17.35}$$

The initial condition is satisfied if $a_0 = 1$.

(**e**) For $n = 1$ Eq. (17.14) reads

$$\left[\frac{\Delta\sigma(\mathbf{x}, \mathbf{x}')}{2} - 2\right] A_1(\mathbf{x}, \mathbf{x}') + \nabla^\mu\sigma(\mathbf{x}, \mathbf{x}')\nabla_\mu A_1(\mathbf{x}, \mathbf{x}') - \Delta A_0(\mathbf{x}, \mathbf{x}') = 0. \tag{17.36}$$

In the coincident limit, we obtain

$$\left[\lim_{\mathbf{x}'\to\mathbf{x}} \frac{\Delta\sigma(\mathbf{x}, \mathbf{x}')}{2} - 1\right] a_1(\mathbf{x}) + \lim_{\mathbf{x}'\to\mathbf{x}} \nabla^\mu\sigma(\mathbf{x}, \mathbf{x}')\nabla_\mu A_1(\mathbf{x}, \mathbf{x}') - \lim_{\mathbf{x}'\to\mathbf{x}} \Delta A_0(\mathbf{x}, \mathbf{x}') = 0. \tag{17.37}$$

Applying the hint, we can write this as

$$a_1(\mathbf{x}) = \lim_{\mathbf{x}'\to\mathbf{x}} \Delta A_0(\mathbf{x}, \mathbf{x}'). \tag{17.38}$$

(**f**) Acting with Δ on Eq. (17.14) for $n = 0$, we obtain

$$0 = \left[\frac{\Delta\sigma(\mathbf{x}, \mathbf{x}')}{2} - 2\right] \Delta A_0(\mathbf{x}, \mathbf{x}') + \frac{\Delta^2\sigma(\mathbf{x}, \mathbf{x}')}{2} A_0(\mathbf{x}, \mathbf{x}') + 2\nabla^\nu\nabla^\mu\sigma(\mathbf{x}, \mathbf{x}')\nabla_\nu\nabla_\mu A_0(\mathbf{x}, \mathbf{x}') + \text{odd terms}, \tag{17.39}$$

where the "odd terms" are terms with an odd number of derivatives of the world function, which vanish in the limit $\mathbf{x}' \to \mathbf{x}$. In the coincidence limit, we therefore obtain

$$\lim_{\mathbf{x}' \to \mathbf{x}} \Delta A_0(\mathbf{x}, \mathbf{x}') = \frac{1}{12} g^{\mu\nu} g^{\rho\sigma} (R_{\rho\mu\sigma\nu} + R_{\rho\nu\sigma\mu}) = \frac{R}{6}. \tag{17.40}$$

Then, by Eq. (17.38), we obtain

$$a_1 = \frac{R}{6}. \tag{17.41}$$

(**g**) By Eq. (17.6), the first corrections to the effective action read

$$\Gamma^{(1)}[g] = -\frac{1}{2} \int_0^\infty \frac{\mathrm{d}s}{(4\pi)^2 s^3} e^{-sm^2} \left(1 + \frac{sR}{6}\right). \tag{17.42}$$

Both terms appearing here are divergent in the limit $s \to 0$. The first term is quadratically divergent in s, *i. e.* quartically divergent in momenta. This is just a constant, thus contributing to the renormalization of the cosmological constant. The second term is linearly divergent in s, so quadratically divergent in momenta. This term is proportional to the Einstein-Hilbert action, thus contributing to the renormalization of the gravitational constant.

(**h**) The additional terms lead to the contribution

$$\Gamma^{(1)}[g] = -\frac{1}{120} \int_0^\infty \frac{\mathrm{d}s}{(4\pi)^2 s} e^{-sm^2} \int \mathrm{d}^4x \sqrt{g} \left(\frac{R^2}{2} + R_{\mu\nu}R^{\mu\nu}\right). \tag{17.43}$$

To renormalize the integral, we integrate from ϵ to infinity as indicated in the hint such that

$$\Gamma^{(1)}_\epsilon[g] = -\frac{1}{120} \int_\epsilon^\infty \frac{\mathrm{d}s}{(4\pi)^2 s} e^{-sm^2} \int \mathrm{d}^4x \sqrt{g} \left(\frac{R^2}{2} + R_{\mu\nu}R^{\mu\nu}\right), \tag{17.44}$$

$$= -\frac{1}{120(4\pi)^2} \int \mathrm{d}^4x \sqrt{g} \left(\frac{R^2}{2} + R_{\mu\nu}R^{\mu\nu}\right) \int_\epsilon^\infty \frac{\mathrm{d}s}{s} e^{-sm^2}, \tag{17.45}$$

$$= -\frac{1}{120(4\pi)^2} \int \mathrm{d}^4x \sqrt{g} \left(\frac{R^2}{2} + R_{\mu\nu}R^{\mu\nu}\right) \Gamma(0, m^2\epsilon), \tag{17.46}$$

$$= \left[\log\epsilon + (\gamma_\mathrm{E} + 2\log m) + \mathcal{O}(\epsilon^2)\right] \frac{1}{120(4\pi^2)} \int \mathrm{d}^4x \sqrt{g} \left(\frac{R^2}{2} + R_{\mu\nu}R^{\mu\nu}\right), \tag{17.47}$$

where $\Gamma(x, a)$ is the incomplete Gamma function. Thus, the renormalized effective action reads

$$\Gamma^{(1)}_\mathrm{ren}[g] = \frac{\gamma_\mathrm{E} + 2\log m}{120(4\pi^2)} \int \mathrm{d}^4x \sqrt{g} \left(\frac{R^2}{2} + R_{\mu\nu}R^{\mu\nu}\right). \tag{17.48}$$

In the limit $m \to 0$, we find an IR divergence. This is usual when integrating out massless particles.

Exercise 18: Palatini gravity

Motivation: Contrary to what you learn in your general-relativity class, general relativity does not have to be a theory of a metric with compatible connection. On shell one can define an equivalent theory in terms of the vielbein, and the connection need not be assumed compatible either. Let's see how, and on the way learn how to juggle with the spin connection!

Given some metric $g_{\mu\nu}$, we can define the tetrad e^a_μ in the usual way, namely as

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu. \tag{18.1}$$

The tetrad implements a local symmetry, namely local Lorentz invariance. The Lorentz transformations act on the Latin indices. To comply with the local symmetry, we have to introduce a covariant derivative on objects carrying local Lorentz indices, say

$$D_\mu V_a = \partial_\mu V_a - \tilde{\omega}_{\mu a}{}^b V_b, \qquad D_\mu V^a = \partial_\mu V^a + \tilde{\omega}_\mu{}^a{}_b V^b, \tag{18.2}$$

with the spin-connection coefficients $\tilde{\omega}_{\mu a}{}^b$. This covariant derivative may also be nontrivial when acting on spacetime vectors

$$D_\mu V^\nu = \partial_\mu V^\nu + \tilde{\Gamma}^\nu_{\mu\rho} V^\rho, \tag{18.3}$$

with some spacetime connection $\tilde{\Gamma}^\rho{}_{\mu\nu}$.

(a) Implement local Lorentz invariance by imposing that

$$D_\mu \eta_{ab} = 0. \tag{18.4}$$

Show that this implies that $\omega_{\mu(ab)} = 0$. Make sure you understand why Eq. (18.4) implies local Lorentz invariance.

(b) While Eq. (18.4) looks a bit like a metric-compatibility condition, it is not. Show that metric compatibility can only be derived from local Lorentz invariance if we enforce tetrad compatibility

$$D_\mu e^a_\nu = 0. \tag{18.5}$$

We do not want to impose any relation between the connection and the vielbein/metric from the start here, so we do not enforce tetrad compatibility from now on.
We define the curvature of the connection $\Omega_{\mu\nu a}{}^b$ as usual for gauge theories with the slight addition that there can be torsion $T^\rho{}_{\mu\nu}$, *i. e.*

$$[D_\mu, D_\nu] V^a = \Omega_{\mu\nu}{}^a{}_b V^b - T^\rho{}_{\mu\nu} \partial_\rho V^a. \tag{18.6}$$

(c) Show that

$$\Omega_{\mu\nu}{}^{ab} = 2\partial_{[\mu} \tilde{\omega}_{\nu]}{}^{ab} + 2\tilde{\omega}_{[\mu}{}^{ac} \tilde{\omega}_{\nu]c}{}^b - T^\rho{}_{\mu\nu} \tilde{\omega}_\rho{}^{ab}, \tag{18.7}$$

$$T^\rho{}_{\mu\nu} = 2\tilde{\Gamma}^\rho_{[\mu\nu]}. \tag{18.8}$$

Let us from now on set the torsion equal to 0, *i. e.* assume a torsion-free connection. Given the curvature, we can construct the imitation Riemann tensor

$$\tilde{R}_{\mu\nu}{}^{\rho\sigma} = e^\rho_a e^\sigma_b \Omega_{\mu\nu}{}^{ab}, \tag{18.9}$$

the imitation Ricci tensor

$$\tilde{R}_{\mu}{}^{\rho} \equiv \tilde{R}_{\mu\nu}{}^{\rho\nu} = e^{\rho}_{a} e^{\nu}_{b} \Omega_{\mu\nu}{}^{ab}, \tag{18.10}$$

and the imitation Ricci scalar

$$\tilde{R} \equiv \tilde{R}_{\nu}{}^{\nu} = e^{\mu}_{a} e^{\nu}_{b} \Omega_{\mu\nu}{}^{ab}. \tag{18.11}$$

These tensors are analogues of the tensors known from Einstein-Hilbert general relativity, but are not *a priori* related to them because the connection is not yet fixed.
With the imitation Ricci scalar, we construct the Palatini action

$$S_{\mathrm{P}} = \frac{1}{16\pi G} \int \mathrm{d}^4 x e e^{\mu}_{a} e^{\nu}_{b} \Omega_{\mu\nu}{}^{ab}, \tag{18.12}$$

where $e = \det e^{a}_{\mu}$. In the following, we show that this action is on-shell equivalent to general relativity under mild assumptions on the connection. The idea is to vary with respect to both the vielbein and the connection.

(d) Vary the action with respect to the vielbein to obtain the equation of motion

$$\tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{R} = 0. \tag{18.13}$$

This looks very much like general relativity already. However, $\tilde{R}_{\mu\nu} \neq R_{\mu\nu}$ in general.

(e) Vary the action with respect to the connection to obtain the constraint

$$D_{\mu} \left(e e^{\mu}_{[a} e^{\nu}_{b]} \right) = 0. \tag{18.14}$$

Hint: Note that while $\tilde{\omega}_{\mu}{}^{ab}$ is not a tensor, $\delta\tilde{\omega}_{\mu}{}^{ab}$ is a tensor, so it makes sense to take its covariant derivative.

(f) Show that Eq. (18.14) implies that the nonmetricity

$$Q_{\rho\mu\nu} = D_{\rho} g_{\mu\nu} \tag{18.15}$$

is constrained to be totally symmetric, *i. e.* such that $Q_{\rho\mu\nu} = Q_{(\rho\mu\nu)}$. Thus, consistency requires the connection to be very close to being the Levi-Civita connection for which $Q_{\rho\mu\nu} = 0$.

Hint: You can use without proof that

$$e\, e^{\mu}_{[a} e^{\nu}_{b]} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} e^{c}_{\rho} e^{d}_{\sigma}. \tag{18.16}$$

(g) If we assume that the connection is the Levi-Civita connection, the theory is equivalent to general relativity at the classical level. Discuss whether it is also equivalent at the quantum level.

(a) The covariant derivative of the Minkowski metric reads

$$D_{\mu} \eta_{ab} = \partial_{\mu} \eta_{ab} - \omega_{\mu a}{}^{c} \eta_{cb} - \omega_{\mu b}{}^{c} \eta_{ac}, \tag{18.17}$$

$$= -\omega_{\mu(ab)}. \tag{18.18}$$

Its vanishing thus requires the spin connection to be antisymmetric. That the covariant derivative of the local Minkowski metric vanishes implies that the Minkowski inner product, necessary for local Lorentz invariance, is preserved under parallel transport. Indeed, parallel transport for a vector V^a requires $D_\mu V^a = 0$, so

$$D_\mu(V_a W^a) = V^a W^b D_\mu \eta_{ab} = 0 \tag{18.19}$$

iff $D_\mu \eta_{ab} = 0$.

(**b**) Metric compatibility implies

$$D_\rho g_{\mu\nu} = D_\rho(\eta_{ab} e^a_\mu e^b_\nu) = (D_\rho \eta_{ab}) e^a_\mu e^b_\nu + \eta_{ab} D_\rho (e^a_\mu e^b_\nu) = 0. \tag{18.20}$$

If we assume local Lorentz invariance, we obtain

$$D_\rho(e^a_\mu e^b_\nu) = 0, \tag{18.21}$$

which implies $D_\rho e^a_\mu = 0$.

(**c**) From the commutator of covariant derivatives, we obtain

$$[D_\mu, D_\nu]V^a = 2D_{[\mu}\left(\partial_{\nu]} V^a + \tilde{\omega}_{\nu]}{}^a{}_b V^b\right), \tag{18.22}$$

$$= 2\left(\partial_{[\mu}\tilde{\omega}_{\nu]}{}^a{}_b + \tilde{\omega}_{[\mu}{}^a{}_{|c|}\tilde{\omega}_{\nu]}{}^c{}_b - \tilde{\Gamma}^\rho{}_{[\mu\nu]}\tilde{\omega}_\rho{}^a{}_b\right) V^b - 2\tilde{\Gamma}^\rho{}_{[\mu\nu]}\partial_\rho V^a. \tag{18.23}$$

Thus, we obtain the curvature and torsion tensors

$$\Omega_{\mu\nu}{}^{ab} = 2\left(\partial_{[\mu}\tilde{\omega}_{\nu]}^{ab} + \tilde{\omega}_{[\mu}{}^{ac}\tilde{\omega}_{\nu]}^{cb}\right) - T^\rho{}_{\mu\nu}\tilde{\omega}_\rho{}^{ab}, \tag{18.24}$$

$$T^\rho{}_{\mu\nu} = 2\tilde{\Gamma}^\rho{}_{[\mu\nu]}. \tag{18.25}$$

(**d**) We have to vary the determinant of the vielbein with respect to the vielbein. Let us use

$$\delta e = \delta\left(\det e^\mu_a\right)^{-1} = -\left(\det e^\mu_a\right)^{-2}\delta\exp(\mathrm{tr}\log e^\mu_a) = -\left(\det e^\mu_a\right)^{-2} e^{\mathrm{tr}\log e^a_\mu}\delta\mathrm{tr}\log e^\mu_a, \tag{18.26}$$

$$= -\left(\det e^\mu_a\right)^{-1}\mathrm{tr}\delta\log e^\mu_a = -e\,\mathrm{tr}\left(e^a_\mu \delta e^\mu_a\right) = -e e^a_\mu \delta e^\mu_a. \tag{18.27}$$

The rest of the variation is just an application of the product rule such that

$$\frac{\delta S_{\mathrm{P}}}{\delta e^\mu_a} = e\left[e^\nu_b(\Omega_{\mu\nu}{}^{ab} + \Omega_{\nu\mu}{}^{ba}) - e^a_\mu \tilde{R}\right] = 0. \tag{18.28}$$

Local Lorentz invariance implies that the connection is antisymmetric in its Lorentz indices, which, in turn, implies that the curvature tensor is antisymmetric in both the Lorentz and the spacetime indices. Thus, it is symmetric in changing both of them at the same time such that we obtain

$$e^\nu_b \Omega_{\mu\nu}{}^{ab} - \frac{1}{2}e^a_\mu \tilde{R} = 0. \tag{18.29}$$

Contracting with e^μ_a, we obtain

$$\tilde{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\tilde{R} = 0. \tag{18.30}$$

While one might get the impression that the tetradic version of Einstein's equations, Eq. (18.29), is actually more constraining than Eq. (18.13), this is not the case because the vielbein is not a fixed external field, so we contracted with any possible vielbein.

(**e**) We have to vary the curvature with respect to the connection to obtain

$$\delta\Omega_{\mu\nu}{}^{ab} = 2\partial_{[\mu}\delta\tilde{\omega}_{\nu]}^{ab} + 2\tilde{\omega}_{[\mu}^{ac}\delta\tilde{\omega}_{\nu]c}{}^b + 2\left(\delta\tilde{\omega}_{[\mu}^{ac}\right)\tilde{\omega}_{\nu]c}{}^b, \tag{18.31}$$

$$= 2\partial_{[\mu}\delta\tilde{\omega}_{\nu]}^{ab} + 2\tilde{\omega}_{[\mu}^{ac}\delta\tilde{\omega}_{\nu]c}{}^b + 2\tilde{\omega}_{[\mu}{}^b{}_{|c|}\delta\tilde{\omega}_{\nu]}^{ac}, \tag{18.32}$$

$$= D_{[\mu}\delta\tilde{\omega}_{\nu]}^{ab}. \tag{18.33}$$

Thus, varying the action, we obtain

$$\frac{\delta S_{\mathrm{P}}}{\delta \tilde{\omega}_{\nu}{}^{ab}} = -2D_{\mu}\left(e e^{\mu}_{[a} e^{\nu}_{b]}\right) = 0. \tag{18.34}$$

(**f**) We can rewrite Eq. (18.14) as

$$D_{\mu}(e e^{\mu}_{[a} e^{\nu}_{b]}) = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd} D_{\mu}(e^{c}_{\rho} e^{d}_{\sigma}) = 0, \tag{18.35}$$

with the epsilon tensors ϵ_{abcd} and $\epsilon^{\mu\nu\rho\sigma}$. As the epsilon tensors are invertible over antisymmetric tensor spaces, they can be inverted to yield

$$D_{[\mu} e^{a}_{\nu]} = 0. \tag{18.36}$$

We need the antisymmetrization here because in Eq. (18.35) we only consider the antisymmetric parts of the covariant derivatives as indicated by the epsilon tensors. This implies that

$$Q_{[\rho\mu]\nu} = D_{[\rho} g_{\mu]\nu} = 0. \tag{18.37}$$

At the same time, the nonmetricity generically satisfies $Q_{\rho\mu\nu} = Q_{\rho(\mu\nu)}$ from the symmetry of the metric, so it is totally symmetric

$$Q_{\rho\mu\nu} = Q_{(\rho\mu\nu)}. \tag{18.38}$$

(**g**) At the quantum level, we care not only about the equations of motion, but also about the measure of the path integral. This measure yields the configuration space, over which we integrate. In general, when using different dynamical variables, this measure is different. Thus, two theories with equivalent actions, but different dynamical variables are not necessarily equivalent. We'll see more about this when considering the Weyl anomaly next week.