Quantum field theory in curved spacetime

Assignment 9/Exam 2 – June 30

Please hand in this assignment before the tutorial at 14h15AM on June 30. In total, you need to obtain 40% of the combined points from the first and this exam.

Exercise 19: Euler-Heisenberg Lagrangian on a background of constant curvature

Motivation: Two weeks ago, we computed the full Euler-Heisenberg Lagrangian on a flat background. This time, we include background curvature and use heat kernels as we learned last week, and only consider the first corrections in curvature and electromagnetic field. We'll find that nonminimal coupling of electromagnetism to gravity is unavoidable!

The Euler-Heisenberg effective action on a curved background satisfies

$$e^{i\Gamma[A,g]} = e^{iS_{EH}[A,g]} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS_{\mathrm{QED}}[A,g]}, \tag{19.1}$$

with the QED action on a curved background and the Einstein-Hilbert action

$$S_{\mathrm{QED}} = \int \mathrm{d}^4x\sqrt{-g}\left[-\mathcal{F} + \bar{\psi}\left(i\not{D} - m\right)\psi\right], \tag{19.2}$$

$$S_{\mathrm{EH}} = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left(R - 2\Lambda\right), \tag{19.3}$$

respectively. Here, $\mathcal{F} \equiv F_{\mu\nu}F^{\mu\nu}/4$, where $F_{\mu\nu}$ is the field-strength tensor of the gauge field A_μ, and R and Λ denote the Ricci scalar and the cosmological constant. Besides, $\not{D}$ denotes the covariant Dirac operator involving the covariant derivative including the spin connection $\omega_{\mu ab}$

$$D_\mu\psi = (\nabla_\mu + ieA_\mu)\psi = \left(\partial_\mu + ieA_\mu - i\omega_{\mu ab}\Sigma^{ab}\right)\psi, \tag{19.4}$$

where $\Sigma^{ab} = i[\gamma^a, \gamma^b]/8$ (note the different notation to exercise 16 where we used $\sigma^{ab} = i[\gamma^a, \gamma^b]/2$), local Lorentz indices are Latin, spacetime indices Greek letters, and e and m are the charge and the mass of the (Grassmann-valued) fermion ψ, respectively. For simplicity, we assume the background electromagnetic field strength to be constant as two weeks ago.

(a) Discuss why a constant field strength necessarily means $\partial_c F_{ab} = 0$, *not* $\partial_\rho F_{\mu\nu} = 0$.

We want to compute the one-loop effective Lagrangian by integrating out the fermion. From exercise 16, we know that we can express the one-loop contribution to the effective action of the background fields as

$$\Gamma^{(1)}[A,g] = -\frac{i}{2}\log\det(\not{D}^2 + m^2), \tag{19.5}$$

$$= \frac{i}{2}\int_0^\infty \frac{\mathrm{d}s}{s} e^{-sm^2}\,\mathrm{tr}\left(e^{-sH}\right) + \mathrm{const}, \tag{19.6}$$

for the Hamiltonian $H = \not{D}^2$.

(**b**) Two weeks ago, we just ignored the (infinite) constant contributions to the effective action. Discuss whether we can still do that. What do they contribute to? Hereafter, we assume to have dealt with those contributions successfully.

(**c**) Demonstrate that the Hamiltonian can be expressed as

$$H = D^2 + 2eF_{ab}\Sigma^{ab} - \frac{R}{4}. \tag{19.7}$$

Hint: You can use without proof that the Lorentz generators satisfy the Lorentz algebra

$$[\Sigma^{ab}, \Sigma^{cd}] = i\left(\eta^{c[a}\Sigma^{b]d} - \eta^{d[a}\Sigma^{b]c}\right). \tag{19.8}$$

Besides the Riemann tensor with Lorentz indices $R_{abcd} = e^\mu_a e^\nu_b R_{\mu\nu cd}$ has the same symmetries as the usual Riemann tensor, and satisfies the first Bianchi identity.

(**d**) Discuss why we can separate the traces as

$$\Gamma^{(1)}[A, g] = \frac{i}{2}\int_0^\infty \frac{\mathrm{d}s}{s} e^{-sm^2} \mathrm{tr}\left(e^{-sH_{\text{kin}}}\right) \mathrm{tr}\left(e^{-sH_{\text{spin}}}\right), \tag{19.9}$$

with $H_{\text{kin}} = D^2 - \frac{R}{4}$, and $H_{\text{spin}} = 2eF_{ab}\Sigma^{ab}$ We have already computed the spin trace in flat space. Explain why this result remains valid and we can copy it to get

$$\mathrm{tr}\left(e^{-sH_{\text{spin}}}\right) = 4\cos(esa)\cosh(esb), \tag{19.10}$$

where

$$a^2 = \sqrt{\mathcal{F}^2 + \mathcal{G}^2} - \mathcal{F}, \qquad b^2 = \sqrt{\mathcal{F}^2 + \mathcal{G}^2} + \mathcal{F}. \tag{19.11}$$

As in exercise 16, here we defined $\mathcal{G} \equiv F_{ab}\tilde{F}^{ab}/4$, and the dual field strength $\tilde{F}^{ab} = \epsilon^{abcd}F_{cd}/2$. Let me remind you that $a = 0$ if $\vec{E} = 0$ and $b = 0$ if $\vec{B} = 0$, so we can understand a to largely measure electric contributions to the field strength, while b largely measures the magnetic ones.

Time to compute $\mathrm{tr}e^{-sH_{\text{kin}}}$. Here, we get to the promised heat kernels. The cool thing about the heat-kernel method is that it does not only work with the Levi-Civita connection; any gauge connection will do. Like last week, we define

$$\frac{\mathrm{d}}{\mathrm{d}s} K(\mathbf{x}, \mathbf{x}', s) = -\left(\mathcal{D}^2 + \mathcal{E}\right) K(\mathbf{x}, \mathbf{x}', s), \tag{19.12}$$

where for the moment $\mathcal{D}_\mu = \partial_\mu + i\mathcal{A}_\mu$ is some differential operator with a gauge connection $\mathcal{A}_\mu$ possibly having suppressed internal indices, while $\mathcal{E}$ is a so-called endomorphism – basically a possibly matrix-valued function of the position like a potential, the Ricci scalar or H_{spin} (above we took that out of the trace, but we can't do so for inhomogeneous electromagnetic fields). Under a gauge transformation $U(x)$, the covariant derivative, the endomorphism and the metric generally transform

$$\mathcal{D}_\mu \to \mathcal{D}^U_\mu = U\mathcal{D}_\mu U^{-1}, \qquad \mathcal{E} \to \mathcal{E}_U = U\mathcal{E}U^{-1}, \qquad g^{\mu\nu} \to g^{\mu\nu}_U = Ug^{\mu\nu}U^{-1}. \tag{19.13}$$

If the gauge transformation concerns internal degrees of freedom, *i. e.* for all gauge transformations but diffeomorphisms, $[g^{\mu\nu}, U] = 0$ such that $g^{\mu\nu}_U = g^{\mu\nu}$.

(**e**) Show that the operator $K_U = UKU^{-1}$ satisfies the heat equation with respect to the transformed operator $(\mathcal{D}^U)^2 + \mathcal{E}^U$. Discuss why this implies that $\int \sqrt{-g}\mathrm{d}^4 x K(x,x,s)$ is a gauge invariant quantity.

(**f**) Let's go back to the specific case in this exercise. We make the ansatz

$$K(x,x,s) = \frac{1}{(4\pi s)^2}\sum_{n=0}^{\infty} a_n(x)s^n. \tag{19.14}$$

To order $n = 1$, we already computed the coefficients $a_0 = 1$, $a_1 = R/6$ last week. Explain why there can't be any new contributions from electromagnetism to that order.

(**g**) Write down the contributions containing the electromagnetic field strength you expect to appear in a_2 and a_3. Explain what those contributions mean.

Hint: Follow the concept that everything that is not prohibited is compulsory.

(**h**) If electromagnetism is nonminimally coupled to gravity, there are a lot of new effects – for example, photons need not move on null geodesics any more, they may perceive an effective spacetime, which is not just governed by $g_{\mu\nu}$. Explain why we do not measure these effects even though they are presumably present.

(a) It is impossible to set $\partial_\rho F_{\mu\nu} = 0$ in general because this equation is coordinate-dependent. However, we are allowed to render a tensor covariantly constant by imposing

$$\nabla_\rho F_{\mu\nu} = 0. \tag{19.15}$$

With a tetrad-compatible connection (as Levi-Civita), we can write this as

$$e^\mu_c \nabla_\mu F_{ab} = e^\mu_c \partial_\mu F_{ab} = \partial_c F_{ab} = 0, \tag{19.16}$$

which implies F_{ab} =const.

(**b**) Infinite constants are multiples of the volume of spacetime V. In the Einstein-Hilbert action, the cosmological-constant term reads

$$S_{\mathrm{CC}} = -\frac{1}{8\pi G}\int \mathrm{d}^4 x\sqrt{-g}\Lambda = -\frac{\Lambda V}{8\pi G}. \tag{19.17}$$

This is independent of the choice of background fields. Thus, constants in the effective action contribute to the cosmological constant. In particular, infinite constants contribute to its renormalization.

(**c**) We compute directly

$$H = \gamma^a\gamma^b e^\mu_a e^\nu_b D_\mu D_\nu, \tag{19.18}$$

$$= (\eta^{ab} - 4i\Sigma^{ab}) e^\mu_a e^\nu_b D_\mu D_\nu, \tag{19.19}$$

$$= D^2 - 4i\Sigma^{ab} e^\mu_a e^\nu_b D_{[\mu} D_{\nu]}, \tag{19.20}$$

$$= D^2 - 2i\Sigma^{ab} e^\mu_a e^\nu_b [D_\mu, D_\nu], \tag{19.21}$$

where we used the anticommutation relations of the γ-matrices, and the antisymmetry of Σ^{ab}. The commutator of gauge covariant derivatives reads $[D_\mu, D_\nu] = ieF_{\mu\nu} + [\nabla_\mu, \nabla_\nu]$, where the commutator of covariant derivatives equals

$$[\nabla_\mu, \nabla_\nu] = - i \left(2\partial_{[\mu}\omega_{\nu]ab}\Sigma^{ab} - i\omega_{\mu ab}\omega_{\nu cd}[\Sigma^{ab}, \Sigma^{cd}]\right), \tag{19.22}$$

$$= - i \left(2\partial_{[\mu}\omega_{\nu]ab}\Sigma^{ab} - \omega_{\mu ab}\omega_{\nu cd}\left(\eta^{c[a}\Sigma^{b]d} - \eta^{d[a}\Sigma^{b]c}\right)\right), \tag{19.23}$$

$$= - 2i\Sigma^{ab}\left(\partial_{[\mu}\omega_{\nu]ab} + \omega_{\mu a}{}^{c}\omega_{\nu cb}\right), \tag{19.24}$$

$$= - i\Sigma^{ab}R_{\mu\nu ab}. \tag{19.25}$$

Contracting the commutator of covariant derivatives as in Eq. (19.21), we obtain

$$\Sigma^{ab}e^\mu_a e^\nu_b[\nabla_\mu, \nabla_\nu] = - i\Sigma^{ab}\Sigma^{cd}e^\mu_a e^\nu_b R_{\mu\nu cd}, \tag{19.26}$$

$$= - i\Sigma^{ab}\Sigma^{cd}R_{abcd}, \tag{19.27}$$

$$= - \frac{i(\Sigma^{ab}\Sigma^{cd} + \Sigma^{cd}\Sigma^{ab})}{2}R_{abcd}, \tag{19.28}$$

$$= - \frac{i(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc} + i\gamma^5\epsilon^{abcd})}{16}R_{abcd}, \tag{19.29}$$

$$= - \frac{i}{8}R, \tag{19.30}$$

where we used the first Bianchi identity $R_{a[bcd]} = 0$. In total, we obtain the Hamiltonian

$$H = D^2 + 2e\Sigma^{ab}F_{ab} - \frac{R}{4}. \tag{19.31}$$

(**d**) The spin-interaction Hamiltonian is expressed solely in terms of Lorentz-group valued degrees of freedom (it has only Latin indices). In the local Lorentz frame, the field strength is still constant. Therefore, the spin-interaction Hamiltonian is position independent, and commutes with the kinetic Hamiltonian. Therefore, as two weeks ago the trace only applies to spinor space, which in a local Lorentz frame is clearly exactly the same as in flat Minkowski spacetime.

(**e**) First, we need to compute

$$(\mathcal{D}^U)^2 = g^{\mu\nu}_U\mathcal{D}^U_\mu\mathcal{D}^U_\nu = Ug^{\mu\nu}U^{-1}U\mathcal{D}_\mu U^{-1}U\mathcal{D}_\nu U^{-1} = Ug^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu U^{-1} = U\mathcal{D}^2U^{-1}. \tag{19.32}$$

Thus, the operator K_U satisfies

$$\frac{\mathrm{d}}{\mathrm{d}s}K_U = U\frac{\mathrm{d}}{\mathrm{d}s}KU^{-1} = -U(\mathcal{D}^2 + \mathcal{E})KU^{-1} = -U(U^{-1}[(\mathcal{D}^U)^2 + \mathcal{E}]UU^{-1}K_UUU^{-1}) = -[(\mathcal{D}^U)^2 + \mathcal{E}_U]K_U. \tag{19.33}$$

The quantity $\mathrm{tr}K = \int\sqrt{-g}\mathrm{d}^4xK(x,x,s)$ is a trace of a gauge covariant operator. In other words, we can write

$$\mathrm{tr}K_U = \mathrm{tr}(UKU^{-1}) = \mathrm{tr}(U^{-1}UK) = \mathrm{tr}(K). \tag{19.34}$$

Therefore, $\mathrm{tr}K$ is gauge invariant.

(**f**) As it is gauge-covariant and a spacetime scalar, the only possible contributions from electromagnetism have to be fully contracted combinations of field strengths (at least one) plus possibly the curvature tensor and the epsilon tensor. Note that s has units of length^2. To order $n = 0$, one would need a dimensionless object, which we don't have at hand. At order $n = 1$, we would need an object of dimension length^{-2} like the field strength $F_{\mu\nu}$ or the dual field strength $\tilde{F}_{\mu\nu}$. However, these are both antisymmetric. Therefore, they yield 0 when being contracted.

(**g**) The possible contributions to a_2 have to be of dimension energy^4, namely

$$a_2 \sim c_{2,1}F^{\mu\nu}F_{\mu\nu} + c_{2,2}\tilde{F}^{\mu\nu}F_{\mu\nu}, \tag{19.35}$$

for some coefficients $c_{2,1}$ and $c_{2,2}$. These are the ordinary electromagnetic Lagrangian and the topological θ-term $\tilde{F}^{\mu\nu}F_{\mu\nu}$. We cannot couple the (dual) field strength nonminimally at that order because the only possible curvature tensor with two indices is the Ricci tensor, which is symmetric.

The possible contributions to a_3 are of dimension energy^6, yielding

$$a_3 \sim c_{3,1}F^{\mu\nu}F^{\rho\sigma}R_{\mu\nu\rho\sigma} + c_{3,2}F^{\mu\nu}\tilde{F}^{\rho\sigma}R_{\mu\nu\rho\sigma} + c_{3,3}\tilde{F}^{\mu\nu}\tilde{F}^{\rho\sigma}R_{\mu\nu\rho\sigma} + c_{3,4}F^{\mu\nu}F_{\mu\nu}R + c_{3,5}F^{\mu\nu}\tilde{F}_{\mu\nu}R. \tag{19.36}$$

Those are all allowed tensor structures due to the mentioned symmetries. All of these nonminimally couple the photon to gravity.

(**h**) In order to obtain the corresponding contributions to the effective action, we still have to integrate over the proper time with the Gaussian weight e^{-sm^2}. Neglecting, for simplicity, the contribution stemming from the spin coupling (which does not change anything conceptual about the result), these terms become

$$\int \mathrm{d}^4x\sqrt{-g}a_3\int_0^\infty \frac{\mathrm{d}s}{(4\pi)^2}e^{-sm^2} = \int \mathrm{d}^4x\sqrt{-g}\frac{a_3}{(4\pi)^2m^2}. \tag{19.37}$$

To estimate the scaling of these terms, let's choose the lightest fermion coupling to electromagnetism, namely the electron. We know that a_3 contains one power of the Riemann tensor. Using Einstein's equations, we can usually estimate that $R \sim G\rho$, where ρ is some measure of the energy density involved. Thus, the contribution to the effective Lagrangian goes like

$$\frac{F^2\rho}{M_\mathrm{P}^2m^2}, \tag{19.38}$$

where F^2 is either $\mathcal{F}$ or $\mathcal{G}$. In a background field, which is just at the threshold to produce electron-positron pairs (this is already really large in astrophysical contexts), we have $F^2 \sim m^4$ such that the contribution reads

$$\frac{m^2\rho}{M_\mathrm{P}^2}. \tag{19.39}$$

Compare this to the leading contributions to the matter Lagrangian. which simply go like ρ, and relative to which the non-minimal-coupling contributions go like

$$\frac{m^2}{M_\mathrm{P}^2} \sim 10^{-40}. \tag{19.40}$$

Thus, the nonminimal coupling is highly suppressed in the astrophysical context.

Exercise 20: Weyl anomaly for a scalar field

Motivation: In the lecture, we already encountered the conformal anomaly for fermions. Now, we have a look at scalar fields.

Consider a real scalar field ϕ whose dynamics is governed by the action

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{R\phi^2}{6}\right). \tag{20.1}$$

We know from assignment 2 that the action is invariant under the Weyl transformation

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega(x)^2 g_{\mu\nu}, \qquad \phi \to \tilde{\phi} = \Omega(x)^{-1}\phi. \tag{20.2}$$

In this exercise, we want to find out whether the conformal symmetry survives for a quantum scalar field using Fujikawa's method. The quantum scalar field is governed by the path integral

$$Z = e^{i\Gamma} = \int \mathcal{D}\phi e^{iS}. \tag{20.3}$$

As S is invariant, we have to look at the measure $\mathcal{D}\phi$. Let's put our system into a box of length L. Then, the Weyl-invariant Klein-Gordon operator has a discrete spectrum, and we define its eigenfunctions ϕ_n as

$$\left(\Box + \frac{R}{6}\right) \phi_n = \lambda_n \phi_n, \tag{20.4}$$

where $\lambda_n \in \mathbb{R}$. These eigenfunctions are orthonormal and complete, *i. e.*

$$\int \mathrm{d}^4x \sqrt{-g} \phi_n^*(x) \phi_{n'}(x) = L^2 \delta_{nn'}, \tag{20.5}$$

$$\sum_n \phi_n^*(x) \phi_n(x') = L^2 \frac{\delta(x - x')}{\sqrt{g}}. \tag{20.6}$$

Besides, they can be chosen such that they are real, *i. e.* $\phi_n^* = \phi_n$, which we do hereafter. Thus, we can express any field ϕ included in the configuration space as

$$\phi = \sum_n c_n \phi_n \tag{20.7}$$

for some set of coefficients c_n.

(a) Define the path integral measure such that the effective action takes the form

$$\Gamma = \frac{i}{2} \log \det L^2 \left(\Box + \frac{R}{6}\right). \tag{20.8}$$

This is what we want the effective action formally to look like.

(b) Demonstrate that after a Weyl transformation – we choose the ϕ_n such that Eq. (20.2) implies $\phi_n \to \tilde{\phi}_n = \Omega^{-1} \phi_n$ – the eigenfunctions are neither orthogonal nor normalized anymore. Instead, show that the functions $\phi_n^\Omega = \Omega^{-1} \tilde{\phi}_n$ are orthonormal in the Weyl-transformed spacetime.

(c) Consider the expansion

$$\tilde{\phi} \equiv \sum_n c_n^\Omega \phi_n^\Omega. \tag{20.9}$$

Construct the corresponding measure in the Weyl-transformed spacetime and show that the two measures are related by a Jacobian

$$\mathcal{D}\phi = \mathcal{D}\tilde{\phi} \det \left(\frac{\partial c_n}{\partial c_{n'}^\Omega}\right), \tag{20.10}$$

where the determinant is taken over the index space n, n'.

(d) Consider an infinitesimal transformation $\Omega(x) = 1 + \omega(x)$. Compute the Jacobian. You should obtain

$$\frac{\partial c_n}{\partial c_{n'}^\Omega} = \delta_{nn'} - L^{-2} \int \mathrm{d}^4x \sqrt{-g} \omega \phi_n \phi_{n'}. \tag{20.11}$$

(**e**) Define

$$Z^\Omega = e^{i\Gamma^\Omega} \equiv \int \mathcal{D}\phi^\Omega e^{iS}, \tag{20.12}$$

to show that

$$\delta\Gamma \equiv \Gamma^\Omega - \Gamma = i \log\det\left(\frac{\partial c_n}{\partial c^\Omega_{n'}}\right) = -iL^{-2}\sum_n \int \mathrm{d}^4x\sqrt{-g}\omega\phi_n^2. \tag{20.13}$$

(**f**) Recall that the effective action is the quantum equivalent of an ordinary action. Analogously to the derivation of the classical stress-energy tensor from the classical action, one can (you don't need to) show that

$$\delta\Gamma = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\langle T_{\mu\nu}\rangle\delta g^{\mu\nu}. \tag{20.14}$$

Demonstrate that

$$\langle T^\mu_\mu(x)\rangle = iL^{-2}\sum_n \phi_n^2(x). \tag{20.15}$$

(**g**) Discuss what happens if we naively apply the completeness relation, Eq. (20.6), to Eq. (20.15). To resolve this issue will be the subject of the next assignment.

(**a**) By analogy with the derivation of the trace anomaly in the lecture we make the ansatz

$$\mathcal{D}\phi = \mathcal{N}\prod_n \mathrm{d}c_n \tag{20.16}$$

for some normalization constant $\mathcal{N}$. We can rewrite the action as

$$\begin{aligned} S &= -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\phi\left(\Box + \frac{R}{6}\right)\phi, && (20.17)\\ &= -\frac{1}{2}\sum_{n,n'}\lambda_{n'}c_nc_{n'}\int \mathrm{d}^4x\sqrt{-g}\phi_n\phi_{n'}, && (20.18)\\ &= -\frac{L^2}{2}\sum_{n,n'}\lambda_{n'}c_nc_{n'}\delta_{nn'}, && (20.19)\\ &= -\frac{L^2}{2}\sum_n \lambda_n c_n^2. && (20.20)\end{aligned}$$

Then, the partition function reads

$$\begin{aligned} e^{i\Gamma} &= \int\left(\prod_n \mathcal{N}\mathrm{d}c_n\right)e^{-\frac{iL^2}{2}\sum_m \lambda_m c_m^2}, && (20.21)\\ &= \prod_n\left(\mathcal{N}\int \mathrm{d}c_n e^{-\frac{iL^2}{2}\lambda_n c_n^2}\right), && (20.22)\\ &= \prod_n\left(\mathcal{N}\frac{\sqrt{-2\pi i}}{L\sqrt{\lambda_n}}\right). && (20.23)\end{aligned}$$

Thus, we can express the effective action as

$$\Gamma = -\, i \log \prod_n \left(\mathcal{N} \frac{\sqrt{-2\pi i}}{L\sqrt{\lambda_n}} \right), \tag{20.24}$$

$$= \frac{i}{2} \log \prod_n \left(\frac{L^2 \lambda_n}{-2\pi i \mathcal{N}^2} \right), \tag{20.25}$$

$$= \frac{i}{2} \log \det \left(L^2 \frac{\Box + \frac{R}{6}}{-2\pi i \mathcal{N}^2} \right), \tag{20.26}$$

where we used that the determinant of an operator amounts to the product of its eigenvalues. If we choose the normalization to be $\mathcal{N} = 1/\sqrt{-2\pi i}$, we indeed obtain

$$\Gamma = \frac{i}{2} \log \det L^2 \left(\Box + \frac{R}{6} \right). \tag{20.27}$$

The corresponding measure reads

$$\mathcal{D}\phi = \prod_n \left(\frac{\mathrm{d}c_n}{\sqrt{-2\pi i}} \right). \tag{20.28}$$

(**b**) Under a Weyl transformation the orthogonality and normalization conditions transform as

$$\int \mathrm{d}^4x \sqrt{-\tilde{g}} \tilde{\phi}_n(x) \tilde{\phi}_{n'}(x) = \int \mathrm{d}^4x \sqrt{-g} \Omega^2(x) \phi_n(x) \phi_{n'}(x) \neq L^2 \delta_{nn'}, \tag{20.29}$$

$$\sum_n \tilde{\phi}_n(x) \tilde{\phi}_n(x') = \sum_n \frac{\phi_n(x)\phi_n(x')}{\Omega(x)\Omega(x')} = L^2 \Omega(x)\Omega(x') \frac{\delta(x-x')}{\sqrt{\tilde{g}}} \neq L^2 \frac{\delta(x-x')}{\sqrt{\tilde{g}}}. \tag{20.30}$$

Instead, the functions ϕ^Ω are orthogonal

$$\int \mathrm{d}^4x (\sqrt{-\tilde{g}}) \phi^\Omega_n(x) \phi^\Omega_{n'}(x) = \int \mathrm{d}^4x \sqrt{-g} \phi_n(x) \phi_{n'}(x) = L^2 \delta_{nn'}. \tag{20.31}$$

and normalized

$$\sum_n \phi^\Omega_n(x) \phi^\Omega_n(x') = \sum_n \frac{\phi_n(x)\phi_n(x')}{\Omega^2(x)\Omega^2(x')} = L^2 \frac{\delta(x-x')}{\Omega^2(x)\Omega^2(x')\sqrt{g}} = L^2 \frac{\delta(x-x')}{\sqrt{\tilde{g}}}. \tag{20.32}$$

(**c**) The analogous measure in the Weyl transformed spacetime reads

$$\mathcal{D}\tilde{\phi} = \prod_n \frac{\mathrm{d}c^\Omega_n}{\sqrt{-2\pi i}}. \tag{20.33}$$

This is an integral over an infinite-dimensional space coordinatized by the c^Ω_n. In getting from $\mathcal{D}\phi$, which is coordinatized by c_n, to $\mathcal{D}\tilde{\phi}$, we have to change coordinates. Such a change of coordinates in an integral is always accompanied by a Jacobian determinant. Thus,

$$\mathcal{D}\phi = \prod_n \frac{\mathrm{d}c_n}{\sqrt{-2\pi i}} = \prod_{n'} \frac{\mathrm{d}c^\Omega_n}{\sqrt{-2\pi i}} \det \left(\frac{\partial c_n}{\partial c^\Omega_{n'}} \right) = \mathcal{D}\tilde{\phi} \det \left(\frac{\partial c_n}{\partial c^\Omega_{n'}} \right). \tag{20.34}$$

(**d**) The Weyl transformation gets us from ϕ to $\tilde{\phi}$. We know that

$$\tilde{\phi} = \sum_n c_n \tilde{\phi}_n. \tag{20.35}$$

At the same time, we know that we can express the conformally transformed field in terms of orthonormal functions as

$$\tilde{\phi} = \sum_n c_n^\Omega \phi_n^\Omega. \tag{20.36}$$

Setting these equal and dividing by Ω, we get

$$\sum_n \frac{c_n \tilde{\phi}_n}{\Omega} = \sum_n c_n \phi_n^\Omega = \sum_n \frac{c_n^\Omega \phi_n^\Omega}{\Omega}. \tag{20.37}$$

Using the orthogonality of ϕ_n^Ω, we obtain

$$c_n = L^{-2} \sum_{n'} c_{n'}^\Omega \int \mathrm{d}^4x \sqrt{\tilde{g}} \Omega^{-1} \phi_{n'}^\Omega(x) \phi_n^\Omega(x). \tag{20.38}$$

For an infinitesimal transformation $\Omega(x) = 1 + \omega(x)$, this becomes

$$c_n = c_n^\Omega - L^{-2} \sum_{n'} c_{n'}^\Omega \int \mathrm{d}^4x \sqrt{\tilde{g}} \omega \phi_{n'}^\Omega(x) \phi_n^\Omega(x). \tag{20.39}$$

Finally, we can take the derivative to obtain

$$\frac{\partial c_n}{\partial c_{n'}^\Omega} = \delta_{nn'} - L^{-2} \int \mathrm{d}^4x \sqrt{\tilde{g}} \omega \phi_{n'}^\Omega(x) \phi_n^\Omega(x), \tag{20.40}$$

$$= \delta_{nn'} - L^{-2} \int \mathrm{d}^4x \sqrt{g} \omega \phi_{n'}(x) \phi_n(x). \tag{20.41}$$

(**e**) While the action depends on the c_n, the Jacobian, Eq. (20.11), does not. Therefore we can write

$$Z^\Omega = \int \mathcal{D}\phi^\Omega e^{iS} = \int \mathcal{D}\tilde{\phi} \det\left(\frac{\partial c_n}{\partial c_{n'}^\Omega}\right)^{-1} e^{iS} = \det\left(\frac{\partial c_n}{\partial c_{n'}^\Omega}\right)^{-1} \int \mathcal{D}\tilde{\phi} e^{iS} = \det\left(\frac{\partial c_n}{\partial c_{n'}^\Omega}\right)^{-1} Z. \tag{20.42}$$

Therefore, we obtain

$$\Gamma^\Omega = -i \log\left[\det\left(\frac{\partial c_n}{\partial c_{n'}^\Omega}\right)^{-1} Z\right] = i \log\det\left(\frac{\partial c_n}{\partial c_{n'}^\Omega}\right) + \Gamma, \tag{20.43}$$

which implies the second equality in Eq. (20.13).

Now we use that $\log\det = \operatorname{tr}\log$ to write

$$\log\det\left(\frac{\partial c_n}{\partial c_{n'}^\Omega}\right) = \sum_n \log\left(1 - L^{-2} \int \mathrm{d}^4x \sqrt{-g} \omega \phi_n \phi_n\right), \tag{20.44}$$

$$\simeq -\sum_n L^{-2} \int \mathrm{d}^4x \sqrt{-g} \omega \phi_n \phi_{n'}, \tag{20.45}$$

where we expanded to first order in the infinitesimal ω. This implies the third equality in Eq. (20.13).

(**f**) The variation of the metric reads in our case

$$\delta g^{\mu\nu} = -2\omega g^{\mu\nu}, \tag{20.46}$$

such that

$$\delta\Gamma = -\int \mathrm{d}^4x \sqrt{-g} \omega \langle T_\mu^\mu \rangle. \tag{20.47}$$

It immediately follows from Eq. (20.13) that

$$\langle T^\mu_\mu \rangle = iL^{-2}\sum_n \phi_n^2(x). \tag{20.48}$$

(**g**) Naively applying the completeness relation to Eq. (20.15), we obtain

$$\langle T^\mu_\mu \rangle = iL^{-2}\frac{\delta(0)}{\sqrt{g}}, \tag{20.49}$$

which is clearly divergent. This sum requires regularization and renormalization.

Extra material 2: Heat-kernel renormalization

Let's try to make sense of Eq. (20.13) which I reprint here

$$\delta\Gamma = -iL^{-2}\sum_n \int \mathrm{d}^4x\sqrt{-g}\omega\phi_n^2. \tag{21.1}$$

We can clearly rewrite

$$\delta\Gamma = -iL^{-2}\sum_n \lim_{s\to 0}\int \mathrm{d}^4x\sqrt{-g}\omega e^{-s\lambda_n}\phi_n^2(x). \tag{21.2}$$

Exchanging sum and limit is generally not allowed. Here, we define the divergent expression Eq. (21.2) via this exchange modulo counterterms X and understand $\phi_n = L\langle x|\lambda_n\rangle$ (position representation of an eigenstate of the Weyl-invariant Klein-Gordon operator)[a] to obtain

$$\delta\Gamma = -iL^{-2}\lim_{s\to 0}\left[\sum_n \int \mathrm{d}^4x\sqrt{-g}\omega\phi_n e^{-s(\Box+R/6)}\phi_n - X\right]. \tag{21.3}$$

Going to Euclidean signature (recall that $\Gamma_\mathrm{E} = -i\Gamma$), we find

$$\delta\Gamma_\mathrm{E} = L^{-2}\lim_{s\to 0}\left[\int \mathrm{d}^4x\sqrt{-g}\omega\sum_n \phi_n e^{s(\Delta+R/6)}\phi_n - X\right]. \tag{21.4}$$

The sum is a heat kernel, and we can use the Seeley-de Witt expansion

$$\sum_n \phi_n e^{s(\Delta+R/6)}\phi_n = \sum_n \langle x|\lambda_n\rangle\langle\lambda_n|e^{s(\Delta+R/6)}|x\rangle = L^2K(x,x,s) = \frac{L^2}{(4\pi s)^2}\sum_{n=0}^{\infty}a_n s^n, \tag{21.5}$$

Clearly, the contributions at orders $n = 0, 1$ are divergent and have to be subtracted off such that

$$X_E = L^2\int \mathrm{d}^4x\sqrt{-g}\omega\frac{a_0 + a_1 s}{(4\pi s)^2}. \tag{21.6}$$

As a result, we obtain

$$\delta\Gamma_\mathrm{E} = \frac{1}{(4\pi)^2}\int \mathrm{d}^4x\sqrt{-g}\omega a_2. \tag{21.7}$$

Going back to Lorentzian signature, the Euclidean heat kernel becomes a Lorentzian heat kernel

$$\delta\Gamma = -\frac{i}{(4\pi)^2}\int \mathrm{d}^4x\sqrt{-g}\omega E_2. \tag{21.8}$$

Now using Eq. (20.14), we finally obtain the trace anomaly

$$\langle T^{\mu}{}_{\mu} \rangle = \frac{i}{(4\pi)^2} E_2(x). \tag{21.9}$$

The heat kernel coefficient E_2 is quadratic in the curvature tensors (Riemann as well as field strength tensors). This result is independent of the size of the box

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[a] Note that we need a factor of L here because $|x\rangle$ has units of $[L^{-2}]$ to make $\int \mathrm{d}^4x\sqrt{-g}|x\rangle\langle x|$ dimensionless, and to satisfy the orthogonality relation, Eq. (20.5).