Condensed Matter Theory problem set 4

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Problem 8: Linear Response

We consider a Hamiltonian $H_S = H_{S,0} + V_S(t)$, where $H_{S,0}$ is the full interacting Hamiltonian and $V_S(t)$ an external perturbation of the form $V_S(t) = F(t)\hat{B}_S$ with $F(t)$ a classical field and $\hat{B}_S$ a (bosonic) quantum mechanical operator. We will measure the reaction of the observable $A_H(t)$ in the Heisenberg picture at time t. The density matrix without perturbation is denoted by $\rho_{S,0} = Z^{-1}\exp(-\beta H_0)$ and the density matrix with perturbation is called $\rho_S(t)$.

Remark: The labels of the operators denote the S = Schrödinger picture, H = Heisenberg picture, and D = Interaction picture (in terms of the "interaction" $V_S(t)$).

- The von-Neumann equation is given by $i\hbar\partial_t\rho_S(t) = [H_S, \rho_S(t)]_-$. Show that the evolution equation of the density matrix in the interaction picture with the boundary condition $\lim_{t\to-\infty}\rho_S(t) = \rho_{S,0}$ is given by

$$\partial_t\rho_D(t) = \frac{i}{\hbar}[\rho_D(t), V_D(t)]_- \,. \tag{1}$$

  Solve equation (1) by iteration.

- Derive the linear response formula for the density matrix by truncating the iteration at first order,

$$\rho_S(t) = \rho_{S,0} - \frac{i}{\hbar}\int_{-\infty}^{t} dt' \exp\left(-\frac{i}{\hbar}H_{S,0}t\right)[V_D(t'), \rho_{S,0}]_- \exp\left(\frac{i}{\hbar}H_{S,0}t\right) . \tag{2}$$

- Use the above result and show that the change of the observable A_H with time can be written as

$$\langle A_H(t)\rangle - \langle A_H(0)\rangle = \frac{1}{\hbar}\int_{-\infty}^{\infty} dt' F(t') G^{\text{ret}}_{AB}(t,t') \tag{3}$$

  with $G^{\text{ret}}_{AB}(t,t') = -i\theta(t-t')\,\mathrm{Tr}\{\rho_{S,0}[A_H(t), B_H(t')]_-\}$ and the Heisenberg picture is determined by $H_{S,0}$. As an explicit example, derive the linear response of the magnetization $\mathbf{M} = \frac{1}{V}\mathbf{m}$ to an external magnetic field $\mathbf{B}(t)$ which is coupled to the magnetic moment $\mathbf{m} = \frac{g_J\mu_B}{\hbar}\sum_i \mathbf{S}_i$.

Problem 9: Diffusion equation in 1d

Solve the one-dimensional diffusion equation

$$\left(\partial_t - D\partial_x^2\right)G^R(xt, x't') = \delta(x-x')\delta(t-t') \tag{4}$$

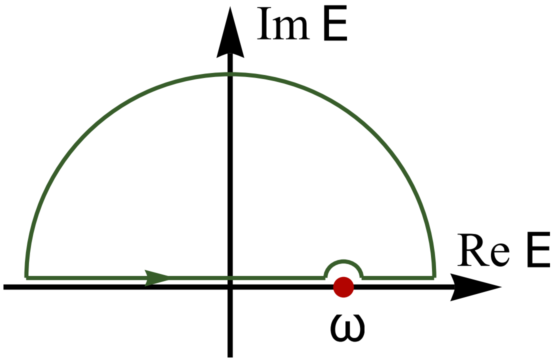
in Fourier space (k, ω). Then Fourier transform to real space (x, t) using contour integration: determine whether you have to close the contour for ω integration above or below so that $e^{-i\omega t}$ is bounded for $t \gtrless 0$; then apply the residue theorem on the diffusion pole in the complex ω plane. You may set $x' = t' = 0$.

Problem 10: Kramers-Kronig relation

The retarded Green's function is analytic in the upper half plane, therefore its real and imaginary parts are connected by the Kramers-Kronig relation

$$\mathrm{Re}\, G^R(k, \omega) = \mathcal{P}\int_{-\infty}^{\infty} \frac{dE}{\pi} \frac{\mathrm{Im}\, G^R(k, E)}{E - \omega} \tag{5}$$

where $\mathcal{P}$ denotes the Cauchy principal value. Derive this identity using a contour that runs along the real line, circumvents the pole at $E = \omega$ and is closed by a semicircle at infinity:



Problem 11: Quasiparticle

With the help of ARPES (*angle-resolved photo-emission spectroscopy*) one can measure the spectral function $A(k_0, E)$ of an electron at a given wavenumber k_0. In one such experiment $A(k_0, E)$ shall have the form of a Lorentz curve with the maximum at energy ϵ_0 and width γ (full width at half height).

(a) What is the properly normalized spectral function $A(k_0, E)$ if the whole spectral weight is contained in this Lorentz peak?

(b) Determine the retarded Green function $G^R(k_0, \omega)$.

[*Hint*: One can, e.g., evaluate the integral in the Lehmann representation as a contour integral. What difference does it make how the contour is closed?]

(c) Compute $G^R(k_0, t)$. How large is the lifetime of the quasiparticle qualitatively (physical interpretation ~ time between scatterings)? Under which condition on ϵ_0 and γ is this a "good" quasiparticle, where one can observe several oscillations in time before it decays?