Condensed Matter Theory — problem set 6

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Problem 15: Grassmann algebra

(a) Compute the integral

$$\int d\eta^* d\eta\, e^{-\eta^* a \eta} \tag{1}$$

for Grassmann numbers η, η^* and $a \in \mathbb{C}$.

(b) The Grassmann δ function is defined as

$$\delta(\xi, \xi') \equiv \int d\eta\, e^{-\eta(\xi - \xi')} \tag{2}$$

for Grassmann numbers ξ, ξ' and η, in analogy to the corresponding expression for complex numbers. Compute the integral in (2) explicitly and show that

$$\int d\xi'\, \delta(\xi, \xi')\, f(\xi') = f(\xi) \tag{3}$$

for an arbitrary Grassmann function $f(\xi) = f_0 + f_1 \xi$ with coefficients $f_0, f_1 \in \mathbb{C}$.

(c) Compute $1/f(\xi)$. Under which condition is this expression well-defined?

(d) For an m-particle Fock state $|m\rangle = c_1^\dagger \dots c_m^\dagger |0\rangle$ and a fermionic coherent state $|\xi\rangle$, derive the identity

$$\langle m|\xi\rangle\langle\xi|m\rangle = \langle -\xi|m\rangle\langle m|\xi\rangle\,. \tag{4}$$

(e) Consider a Grassmann algebra with two generators ξ_1 and ξ_2. Specify a basis for this algebra; which dimension does it have? The basis elements $\{z_i\}$ of the algebra satisfy multiplication rules of the form $z_i \cdot z_j = (M_i)_{kj} z_k$; find an explicit matrix representation for each basis element that satisfies the same rules, and express the general Grassmann function

$$A(\xi_1, \xi_2) = a_0 + a_1 \xi_1 + a_2 \xi_2 + a_{12} \xi_1 \xi_2 \tag{5}$$

as a matrix in this representation.

Problem 16: Perturbation theory

Consider the integral

$$I(g) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}x^2 - gx^4\right\} \tag{6}$$

to mimic a particle with an anharmonic term, or "interaction", of strength g. Expand the integral into a series $I(g) = \sum_n g^n I_n$ for $g > 0$ and show that for large n,

$$g^n I_n \sim \left(-\frac{16gn}{e}\right)^n \tag{7}$$

by expressing the Gaussian integrals $\int dx\, x^{4n}\, e^{-x^2/2}$ in terms of factorials and using the Stirling formula $n! \sim (n/e)^n$. Discuss whether the series converges, and if not, at which order the expansion starts to break down (depending on g). How large is the radius of convergence around $g = 0$ (consider negative g)? Estimate the error of a partial resummation up to order $n_{\max}$,

$$\left| I(g) - \sum_{n=0}^{n_{\max}} g^n I_n \right|. \tag{8}$$

Given g, estimate the value of $n_{\max}$ where the error is minimal. How large is the error at $n_{\max}$?

Problem 17: Landau levels in graphene

The Hamiltonian for graphene close to the K point of the Brillouin zone takes the form of a Dirac Hamilton operator (cf. problem 6):

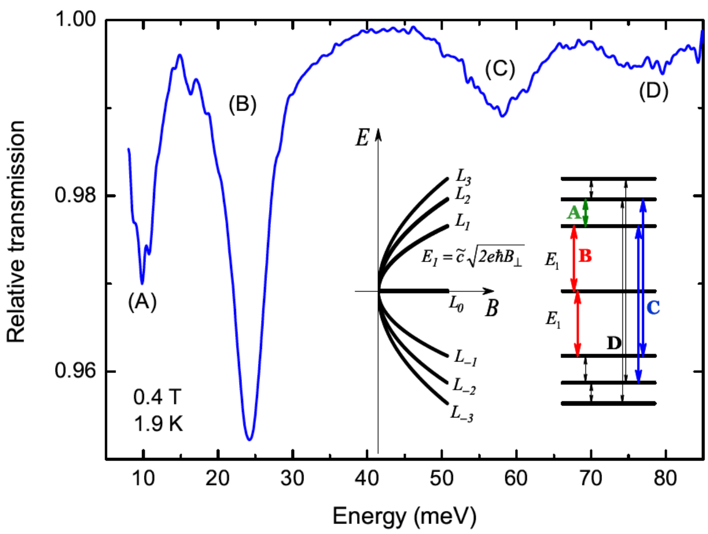
$$\hat{H}_0^K = v_F \boldsymbol{\tau} \cdot \boldsymbol{p}, \quad \text{with} \quad \boldsymbol{p} = (p_x, p_y)^T \quad \text{and the Pauli matrices} \quad \boldsymbol{\tau} = (\tau_x, \tau_y).$$

Electrons in an external magnetic field can be described by replacing the momentum operator by its gauge invariant form $\boldsymbol{p} \to \boldsymbol{P} = \boldsymbol{p} + e\boldsymbol{A}(\boldsymbol{r})$ where $\boldsymbol{B} = \nabla \times \boldsymbol{A}(\boldsymbol{r})$.

(a) Calculate the commutator $[P_x, P_y]$ and show that P_x and P_y are conjugate variables. For this purpose use a vector potential in Landau gauge $\boldsymbol{A}(\boldsymbol{r}) = B(-y, 0, 0)$ and express the result in terms of the *magnetic length* $l_B = \sqrt{\hbar/(eB)}$.

(b) Introduce ladder operators $a, a^\dagger$ as linear combinations of P_x and P_y. They should satisfy $[a, a^\dagger] = 1$. Express the matrix-valued Dirac Hamiltonian in a magnetic field $\hat{H}_B^K$ in terms of ladder operators and the characteristic frequency $\omega = \sqrt{2} v_F / l_B$.

(c) To determine the eigenvalues and eigenstates of the Hamiltonian we want to solve the eigenvalue equation $\hat{H}_B^K \psi_n = \epsilon_n \psi_n$. Here, ψ is a 2-spinor, $\psi_n = (u_n, v_n)^T$. Show that the second spinor component v_n is an eigenstate of the occupation number operators $n = a^\dagger a$, i.e., $v_n \propto |n\rangle$.



(d) Now you can determine the energy eigenvalues ϵ_n as a function of the occupation number. These are the so-called relativistic Landau levels of graphene. Sketch ϵ_n as a function of the magnetic field.

(e) With the solution for v_n you can also determine the solution for u_n. Give an explicit expression for the full spinor $\psi_{\lambda,n}$ and discuss the case $n = 0$. Note that there are positive and negative energy solutions ($\lambda = \pm$).

(f) Relativistic Landau levels can be observed experimentally by transmission spectroscopy. In this method the intensity of the light transmitted by a graphene sample is measured. Monochromatic light induces dipole transitions between Landau levels $n \to n+1$. Calculate the transition energies Δ_n and compare your results with the measurement given in the figure below[1] ($B = 0.4\,\mathrm{T}$).

[1]taken from M. L. Sadowski *et al.*, Phys. Rev. Lett. **97**, 266405 (2006)