Extra Dimensions

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Extra Dimensions

1. Motivation

* Why do we live in 3+1 dimensions?
  → consider other possibilities in order to find answer

  Is there more than we can see?

* Kaluza, Klein: (1919 ... 1926)
  general relativity and $U(1)$ gauge theory can be unified in 5 dimensions!
  (did not work in original form, but maybe other possibilities ?!)
  → unification of gravity and gauge theory might require extra dimensions

* String theory (candidate for quantum gravity!) is consistent only in higher-dimensional space.
  5 consistent superstring theories found:
  type I, IIA, IIB, heterotic $\begin{cases} E_8 \times E_8, \\ SO(32) \end{cases}$
  all only consistent in $d=10$ dimensions

In string theory usually assumed
that 6 dimensions are compactified
on very small size manifold,
but mechanism for this not understood

→ try bottom-up approach!
→ field theory with additional dimensions

* Standard Model (SM) is very unlikely
to be the full story!
SM might work up to Planck scale M_{Pl}
($M_{Pl} \simeq 10^{19}$ GeV), at higher energy
quantum gravity should set in.
More likely appears grand unification
at about $M_{GUT} \simeq 10^{16}$ GeV, as suggested
by running couplings of SM, possibly
with supersymmetry (SUSY) added.
In any case M_{Pl} is natural cut-off
for SM physics.

Problems with SM: too many unknown
parameters
in particular: hierarchy problem!

→ Hierarchy problem:

* In SM the electroweak scale M_{EW} is set by the Higgs potential

$$V(\phi) = -\mu^2 \phi^2 + \lambda \phi^4$$

→ Higgs VEV : $\frac{1}{\sqrt{2}} v$

$$v = \sqrt{\frac{\mu^2}{\lambda}} = 246 \text{ GeV}$$

Higgs mass $m_H = \sqrt{2} \mu$

Quantum corrections to Higgs mass are quadratically divergent:

$$\Delta \mu^2 \sim \Lambda^2_{cutoff}$$

→ If SM is valid up to M_{Pl} an enormous fine-tuning is required ("natural" value for m_H would be $O(M_{Pl})$)

* usual solution to hierarchy problem: SUSY !

with SUSY the corrections

have opposite sign

→ cancellation of quadratic divergences

With SUSY also GUT appears more likely. Still, the large ratio $\frac{M_{GUT}}{M_{EW}}$ remains unexplained.

Possible solution in context of extra dimensions: maybe in higher-dim. theory m_{EW} is the fundamental scale, and our 4d M_{Pl} appears large only because extra dimensions are large!?

* New ideas: (1980's, but mostly since 1998)

Extra dimensions might be much larger than M_{Pl}!

- naively would expect at most

$$r_{extra} \leq TeV^{-1} \simeq 10^{-18} m$$

from direct observation (absence of KK towers, see below)

- But: need not be true
  - * large extra dimensions $R \simeq 0.01$ mm of Arkani-Hamed, Dimopoulos, Dvali (ADD)
  - * warped extra dimensions of Randall and Sundrum (RS) even infinitely large possible

→ new activity in this field,
$\mathcal{O}(10^3) - \mathcal{O}(10^4)$ publications,
due to new possibilities in model building
(model building necessary in order to know what to look for at LHC, ...)
and in attempts to solve problems of the SM (hierarchy problem, SUSY breaking etc)

* important ingredient in most of these theories (models): <u>branes</u>.
Branes are lower-dimensional objects (<u>p-brane</u> has p space-like dimensions) in a higher-dim. space to which particles can be confined, or on which particles can be localized.
In string theory branes (in particular D-branes, Dirichlet branes) appear naturally as solutions of low energy (supergravity) equations. Open strings can end on such branes, hence particle modes of these strings are naturally confined to the brane.

* Variety of possibilities with large and small extra dimensions have been discovered:

  - large extra dimensions (ADD)
  - warped extra dimensions RS I, RS II
  - TeV scale extra dimensions, size $\sim 10^{-18}$ m
    ($\rightarrow$ universal extra dimensions, UED)
  - GUT scale extra dimensions
  - deconstructed extra dimensions
    (dimensions emerging from dynamics of gauge theories)

  Not all address same problems, mechanisms of different models can often be combined

* general picture:

  effective theories in mind

  $\rightarrow$ renormalizability is not a problem

  (5d gauge theories in general not renormalizable)

* These lectures:

  - Introduction to main ideas of extra dimensions, main models, some interesting mechanisms

  - assumed knowledge:
    basics of general relativity
    QFT
    gauge theory, Standard Model
    some cosmology

  - no attempt at historical completeness, only few references

    For further reading (lecture notes, some original papers etc) see
    www.thphys.uni-heidelberg.de/~ewerz

2. Kaluza - Klein (KK) theory

Consider 3+1 - dim spacetime with coordinates

$$x = (x^0, \dots, x^3) = x^\mu$$

and metric $g_{\mu\nu}$, $\mu, \nu = 0, \dots, 3$.

We use sign convention $(+,-,-,-)$ for the metric, that is for flat metric we have

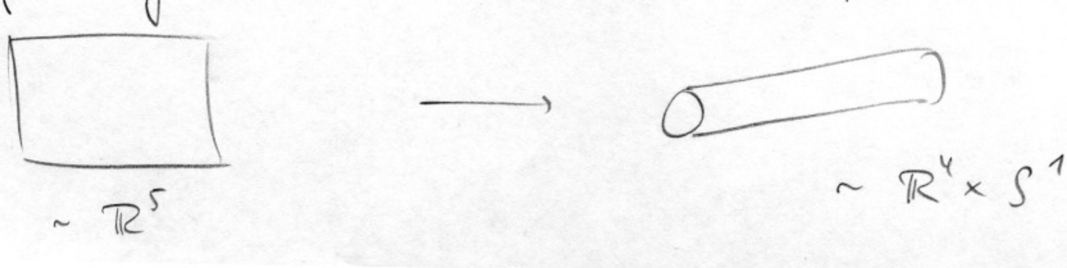
$$\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-).$$

Consider further a D-dimensional space-time of the form

$$M^4 \times X^{D-4},$$

that is a direct product of Minkowski spacetime M^4 with a compact ("internal") manifold X^{D-4} representing some extra dimensions. (We assume $M^4 \times X^{D-4}$ to be a solution of D-dim. Einstein equations.)

A simple example is a 5-dim. spacetime compactified on a circle (= periodic identification)



In the D-dim. spacetime we have coordinates

$$X = (x^0, \dots, x^3, x^5, \dots x^D) = X^A$$

and metric

$$g_{MN} \ (\text{or } G_{MN}) \quad M,N = 0, 1, \dots, 3, 5, \dots, D$$

and we use again sign conventions $(+, -, \dots, -)$.

Let us now consider $M^4 \times S^1$, with

$$x^5 = y \in [0, 2\pi L[$$

with periodic boundary conditions.

* We start with a scalar field ϕ of mass m, 5-d. Lagrangian is

$$\mathcal{L} = \frac{1}{2} \partial_A \phi \, \partial^A \phi - \frac{1}{2} m^2 \phi^2$$

From periodicity in x^5-direction:

$$\phi(x, y) = \phi(x, y + 2\pi L)$$

Expanding in harmonics on the circle:

$$\phi(x,y) = \sum_{n=-\infty}^{\infty} \phi_n(x) \, e^{iny/L}$$

to diagonalize ∂_5. Reality of ϕ requires

$$\phi_n^* = \phi_{-n} .$$

Substituting this in $\mathcal{L}$ we obtain for the action

$$S = \int d^4x \int_0^{2\pi L} dy \; \mathcal{L}$$

with the rescaling $\varphi_n = \sqrt{2\pi L}\, \phi_n$

$$S = \int d^4x \Big[\frac{1}{2} \partial_\mu \varphi_0 \, \partial^\mu \varphi_0 - \frac{1}{2} m^2 \varphi_0^2 + \sum_{n=1}^{\infty} \Big(\partial_\mu \varphi_n \, \partial^\mu \varphi_n^* - \Big(m^2 + \frac{n^2}{L^2}\Big) \varphi_n \varphi_n^* \Big) \Big]$$

Here we can read off the resulting 4d effective theory.

The spectrum is:

- a single real scalar field φ_0, a so-called zero-mode (which does not depend on y in the original 5d theory) – often the term "zero-mode" is used only for massless modes.
  We have mass m for φ_0.
- an infinite tower of massive complex scalars φ_n with masses

$$m_{eff}^2 = m^2 + \frac{n^2}{L^2}$$

These φ_n are called KK modes.

For small m ($m \to 0$) this implies an equidistant mass spectrum, $m_{eff} = \frac{n}{L}$.

More generally, compactification on a torus $T^\delta = \underbrace{S^1 \times \ldots \times S^1}_{\delta \text{ extra dimensions}}$

we find KK modes

$$\varphi_{\vec{n}} \quad \text{with} \quad \vec{n} = (n^5, \ldots, n^{4+\delta}) \in \mathbb{Z}^\delta$$

with masses

$$m_{eff}^2 = m^2 + \left(\frac{\vec{n}}{L}\right)^2 .$$

Let us now turn to the original model of Kaluza and Klein, which shows that 4d Einstein gravity and electrodynamics can be unified in a 5d theory.

Consider 5d Einstein-Hilbert action

$$S = \frac{1}{2} M_*^3 \int d^4x \, dy \, \sqrt{G} R^{(5)}$$

on the space $M^4 \times S^1$.

Here M_* is the fundamental Planck scale in 5d (M_*^3 occurs to make S dimensionless, $\hbar = 1$).

$R^{(5)}$ is the 5d scalar curvature.

[Aside: short (!) summary of general relativity:

metric $g_{\mu\nu}$ on manifold M in 4d

→ obtain Levi-Civita connection with Christoffel symbols $\Gamma^{\sigma}_{\lambda\mu}$ related to $g_{\mu\nu}$ via

$$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left[\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} \right]$$

then curvature tensor $R^{\lambda}{}_{\mu\nu\kappa}$ is

$$R^{\lambda}{}_{\mu\nu\kappa} = \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu} \Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa} \Gamma^{\lambda}_{\nu\eta} .$$

Get Ricci tensor $R_{\mu\kappa}$

$$R_{\mu\kappa} = R^{\lambda}{}_{\mu\lambda\kappa}$$

and scalar curvature R

$$R = g^{\mu\kappa} R_{\mu\kappa}$$

Then Einstein equation is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G_N T_{\mu\nu}$$

$G_N = \dfrac{1}{8\pi M_{Pl}^2}$ energy momentum tensor $T_{\mu\nu}$

Equivalent Einstein-Hilbert action is

$$S = \frac{1}{2} M_{Pl}^2 \int d^4x \sqrt{g}\, R$$

det $g_{\mu\nu}$]

Expand G_{AB} in harmonics on the circle:

$$G_{AB}(x,y) = \sum_{n=-\infty}^{\infty} G_{AB}^{(n)} \, e^{iny/L}$$

For now concentrate only on zero-mode $G_{AB}^{(0)}$ and neglect all massive modes.

We decompose G_{AB} as

$$G_{AB}^{(0)} = \left(\begin{array}{c|c} G_{\mu\nu}^{(0)} & G_{\mu 5}^{(0)} \\ \hline G_{5\mu}^{(0)} & G_{55}^{(0)} \end{array} \right)$$

and write

$$G_{\mu\nu}^{(0)} = e^{\phi/\sqrt{3}} \left[g_{\mu\nu}(x) + e^{-\sqrt{3}\phi} A_\mu A_\nu \right]$$

$$G_{\mu 5}^{(0)} = G_{5\mu}^{(0)} = e^{-2\phi/\sqrt{3}} A_\mu$$

$$G_{55}^{(0)} = e^{-2\phi/\sqrt{3}}$$

That gives for the 4d effective action (with only the zero-mode fields)

$$S_{\text{zero-mode}} = M_*^3 \pi L \int d^4x \sqrt{g} \left[R^{(4)} + \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{4} e^{-\sqrt{3}\phi} F_{\mu\nu}^2 \right]$$

Note that in the above ds_5^2 is invariant under the gauge transformation

$$g_{\mu\nu} \to g_{\mu\nu} \quad , \quad \phi \to \phi \quad , \quad A_\mu \to A_\mu + \partial_\mu \alpha$$

Assuming constant ϕ we thus have 4d gravity and a $U(1)$ gauge field.
(ϕ actually corresponds to size of extra dim., but ϕ does not have a potential → problems with this theory.)
Comparing with conventional 4d Einstein-Hilbert action we find that

$$M_{Pl}^2 = M_*^3 \, 2\pi L$$

or for the Newton constant

$$G_N = \frac{1}{8\pi M_{Pl}^2} = \frac{1}{16\pi^2 L M_*^3} \, .$$

The massive KK levels give a massive graviton with mass $m_n^2 = \frac{n^2}{L^2}$ at each (n'th) level. Their mass comes due to Higgs mechanism: one massless graviton (2 d.o.f.) eats one massless gauge field (2 d.o.f.) and one real scalar (1 d.o.f.) to make a massive 4d graviton (5 d.o.f.).
Massive KK modes are charged under the massless gauge field (charges $\sim q_n \sim \frac{m_n}{M_{Pl}}$).
At linearized level gauge transfs do not mix different KK levels, but do at nonlinear gravity level.

With more complicated internal spaces X^{D-4} one can also obtain non-Abelian gauge theories. However, then X^{D-4} is in general curved, hence Λ-term is required in order that $M^4 \times X^{D-4}$ is solution of Einstein equations. $\rightarrow$ also M^4 curved, hence more complications - not to be discussed here!

Up to energies of ~ 1 TeV no KK towers have been observed.

$\rightarrow$ such a scenario does not allow large extra dimensions.

$\rightarrow$ We next turn to models with branes.

3. Large extra dimensions (ADD model)

1) Simple picture

Recall hierarchy problem: $M_{EW} \ll M_{Pl}$

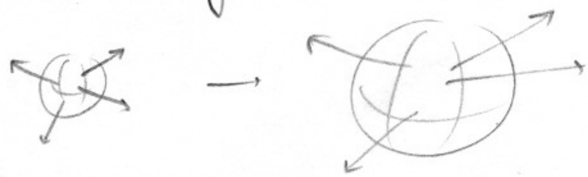
→ Why is gravity so weak?

- electric force between two electrons $\sim 10^{43}$ times stronger than gravitational force!

  These forces would be equal if electron had 10^{22} times more mass, that is had $\sim M_{Pl} = 10^{19}$ GeV. $\hat{=}$ 10^{-35} m.

In 3 space dimensions grav. force is

$F \sim \frac{1}{R^2}$ (since surface $\sim R^2$



)

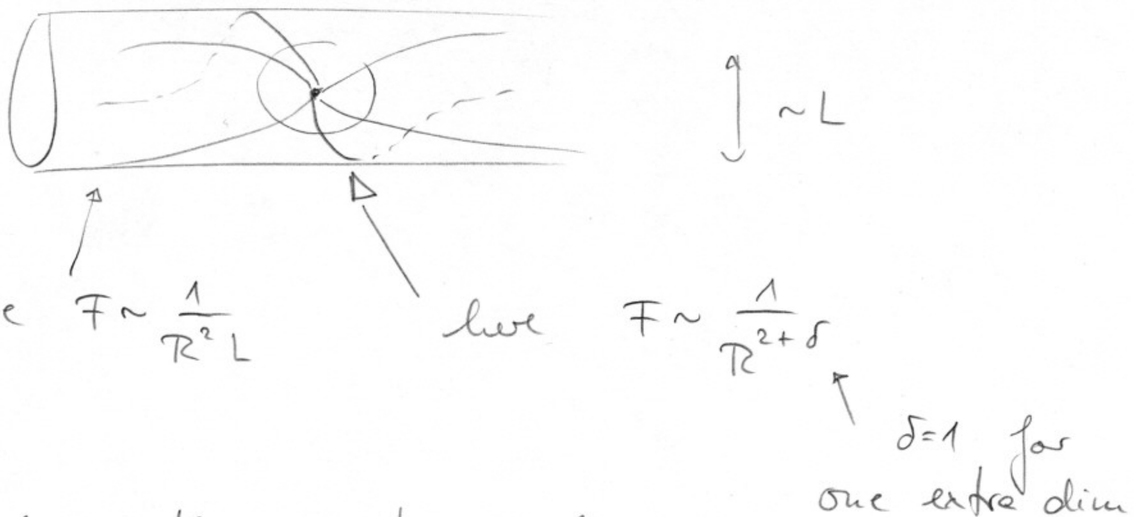
while in 4 space dim.

$F \sim \frac{1}{R^3}$ (surface $\sim R^3$)

→ If gravity has more dimensions to propagate in, force becomes strong already at larger distance.

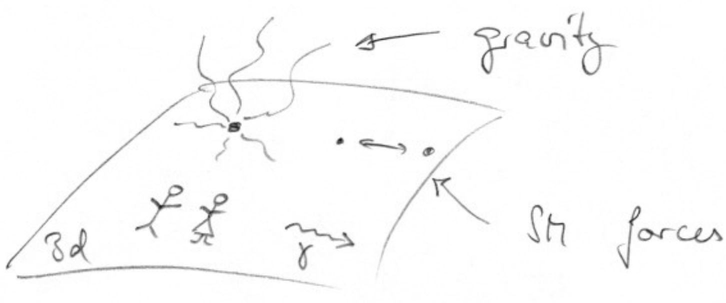
→ unification at M_{EW} instead of M_{Pl} ?!

In a model with one compactified extra dimension this gives

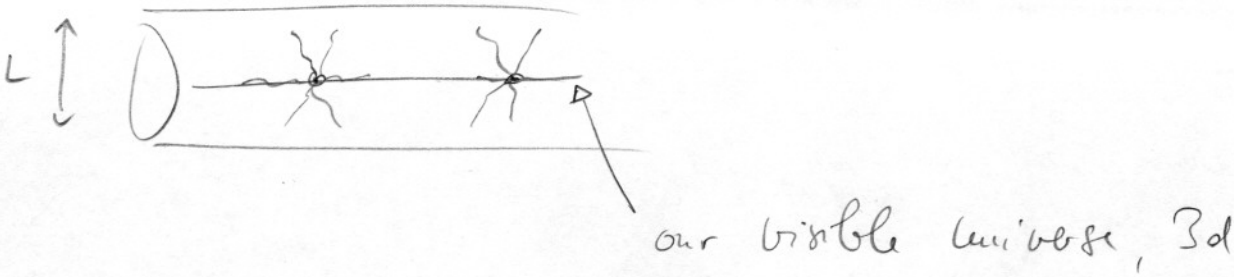


Why does this not apply to electroweak & strong interactions?

→ SM particles and interactions confined to a brane!



or with compactified dim.



2) Matching higher dimensional to 4d theory

How do we obtain the mass scales and couplings in the 4d effective theory from the higher-dimensional fundamental theory?

Consider higher-dim theory with Planck scale M_* and compact extra dimensions of size r. Metric tensor g_{AB} has mass dimension $[g]=0$, Christoffel symbol $[\Gamma]=1$, Ricci tensor and scalar curvature $[R_{MN}]=2$, $[R]=2$, all independent of total number of dimensions.

Einstein-Hilbert action in $4+n$ dimensions:

$$S_{4+n} = -M_*^{n+2} \int d^{4+n}x \sqrt{g^{(4+n)}}\, R^{(4+n)}$$

such that S_{4+n} dimensionless ($\hbar = 1$)

As in example of KK model we have to match this to 4d action

$$S_4 = -M_{Pl}^2 \int d^4x \sqrt{g^{(4)}}\, R^{(4)}$$

For this knowledge of extra-dim. geometry required.

We now assume flat spacetime and n compact extra dimensions, hence

$$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})\, dx^\mu dx^\nu - r^2 d\Omega^2_{(n)}$$

with $h_{\mu\nu}$ 4d fluctuation around flat 4d Minkowski metric $\eta_{\mu\nu}$. This tells us how 4d graviton ($h_{\mu\nu}$) is contained in higher-dim. action.

From this we get

$$\left.\begin{aligned}\sqrt{g^{(4+n)}} &= r^n \sqrt{g^{(4)}} \\ R^{(4+n)} &= R^{(4)}\end{aligned}\right\} \text{ calculated from } h_{\mu\nu}.$$

Hence

$$S_{4+n} = - M_*^{n+2} \underbrace{\int d\Omega_{(n)}\, r^n}_{= V_{(n)} \text{ volume of extra-dim. space}} \int d^4x \sqrt{g^{(4)}}\, R^{(4)} ,$$

for torus compactification

$$V_{(n)} = (2\pi r)^n$$

Hence

$$M_{Pl}^2 = V_{(n)} M_*^{n+2} = (2\pi r)^n M_*^{n+2}$$

→ a large volume of the extra dim. can induce a large M_{Pl} although M_* is not large!

Similarly, the matching for gauge fields can be done. In $(4+n)$ d (gauge fields in all dimensions!)

$$S^{(4+n)} = - \int d^{4+n}x \, \frac{1}{4 g_*^2} F_{MN} F^{MN} \sqrt{g^{(4+n)}}$$

with higher dim. field strength tensor F_{MN} and $(4+n)$ d gauge coupling g_*.
Concentrating on how 4d field strength $F_{\mu\nu}$ are contained in F_{MN}, we can integrate over extra dim.:

$$S^{(4)} = - \int d^4x \, \frac{V_{(n)}}{4 g_*^2} F_{\mu\nu} F^{\mu\nu} \sqrt{g^{(4)}} + \ldots$$

Thus couplings are related by

$$\frac{1}{g_{eff}^2} = \frac{V_{(n)}}{g_*^2} .$$

Note that higher-dim. coupling g_* hence has to have mass dimension (since $[g_{eff}] = 0$)

$$[g_*] = - \frac{n}{2} .$$

Consequently, higher-dim. gauge theory cannot be renormalizable.
→ has to be considered as effective theory of some more fundamental theory at even higher energy (strings?)

Natural assumption is that scale of $(4+n)d$ gauge coupling is also M_*,

$$g_* \sim \frac{1}{M_*^{n/2}}$$

Hence

$$\frac{1}{g_4^2} = V_{(n)} M_*^n \sim r^n M_*^n$$

$$M_{Pl}^2 = V_{(n)} M_*^{n+2} \sim r^n M_*^{n+2}$$

$$\rightarrow \quad r \sim \frac{1}{M_{Pl}} g_4^{-\frac{n+2}{n}}$$

For $g \sim \mathcal{O}(1)$ the "natural" size of the extra dimensions is hence $r \sim \frac{1}{M_{Pl}}$.

This was the general opinion until the 1990's. Note that above relation results from assuming gauge fields to propagate in all dimensions. If they were confined to only 4d then only the strength of gravity (M_{Pl} vs. M_*) would be affected.

3) How large can "large" possibly be?

At distances below the compactification radius we expect deviations from $\frac{1}{r}$ - potential for gravity. Actual experiments can now measure gravity at distances down to ~ 0.1 mm, and so far no deviation has been found. Hence we have the bound

$$r \leq 0.1 \text{ mm}$$

Since $M_*^{n+2} \sim \frac{M_{Pl}^2}{r^n}$, $(4+n)$d Planck scale is lowered by the extra dimensions if $r > \frac{1}{M_{Pl}}$.

$M_* \lesssim 1$ TeV is excluded from the non-observation of strong gravity effects at colliders so far.

→ lowest possible value is ~ 1 TeV, thus

$$M_* \geq 1 \text{ TeV}.$$

Such models with $M_* \sim O(1 \text{ TeV})$ are called "large extra dimensions".

(Arkani-Hamed, Dimopoulos, Dvali, '98, also Antoniadis '90)

Tests of the Gravitational Inverse-Square Law below the Dark-Energy Length Scale

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(Dated: November 30, 2007)

We conducted three torsion-balance experiments to test the gravitational inverse-square law at separations between 9.53 mm and 55 μm, probing distances less than the dark-energy length scale $\lambda_d = \sqrt[4]{\hbar c/\rho_d} \approx 85\ \mu$m. We find with 95% confidence that the inverse-square law holds ($|\alpha| \leq 1$) down to a length scale $\lambda = 56\ \mu$m and that an extra dimension must have a size $R \leq 44\ \mu$m.

PACS numbers: 04.80.-y,95.36.+x,04.80.Cc,12.38.Qk

Recent cosmological observations[1, 2, 3] have shown that 70% of all the mass and energy of the Universe is a mysterious "dark energy" with a density $\rho_d \approx 3.8$ keV/cm^3 and a repulsive gravitational effect. This dark-energy density corresponds to a distance $\lambda_d = \sqrt[4]{\hbar c/\rho_d} \approx 85\ \mu$m that may represent a fundamental length scale of gravity[4, 5]. Although quantum-mechanical vacuum energy should have a repulsive gravitational effect, the observed ρ_d is between 10^{60} to 10^{120} times smaller than the vacuum energy density computed according to the standard laws of quantum mechanics. Sundrum[6] has suggested that this huge discrepancy (the "cosmological constant problem") could be resolved if the graviton were a "fat" object with a size comparable to λ_d that would prevent it from "seeing" the short-distance physics that dominates the vacuum energy. His scenario implies that the gravitational force would *weaken* for objects separated by distances $s \lesssim \lambda_d$. Dvali, Gabadaze and Senjanovíc[7] argue that a similar weakening of gravity could occur if there are extra *time* dimensions. In their scenario, the standard model particles are localized in "our" time, while the gravitons propagate in the extra time dimension(s) as well. Other scenarios predict the opposite behavior: the extra *space* dimensions of M-theory would cause the gravitational force to get *stronger* for $s \lesssim R$ where R is the size of the largest compactified dimension[8]. These considerations, plus others involving new forces from the exchange of proposed scalar or vector particles[9] motivated the tests of the gravitational inverse-square law we report in this Letter.

Our tests were made with a substantially upgraded version of the "missing mass" torsion-balance instrument used in our previous inverse-square-law tests[10, 11]. The instrument used in this work[12], shown in Fig. 1, consisted of a torsion-pendulum detector suspended by a thin $\approx$ 80-cm-long tungsten fiber above an attractor that was rotated with a uniform angular velocity ω by a geared-down stepper motor. The detector's 42 test bodies were 4.767-mm-diameter cylindrical holes machined into a 0.997-mm-thick molybdenum detector ring. The hole centers were arrayed in two circles, each of which had 21-fold azimuthal symmetry. The attractor had a similar 21-fold azimuthal symmetry and consisted of a 0.997

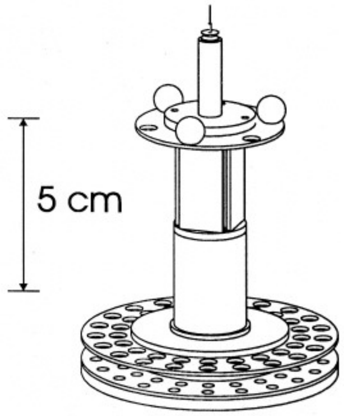


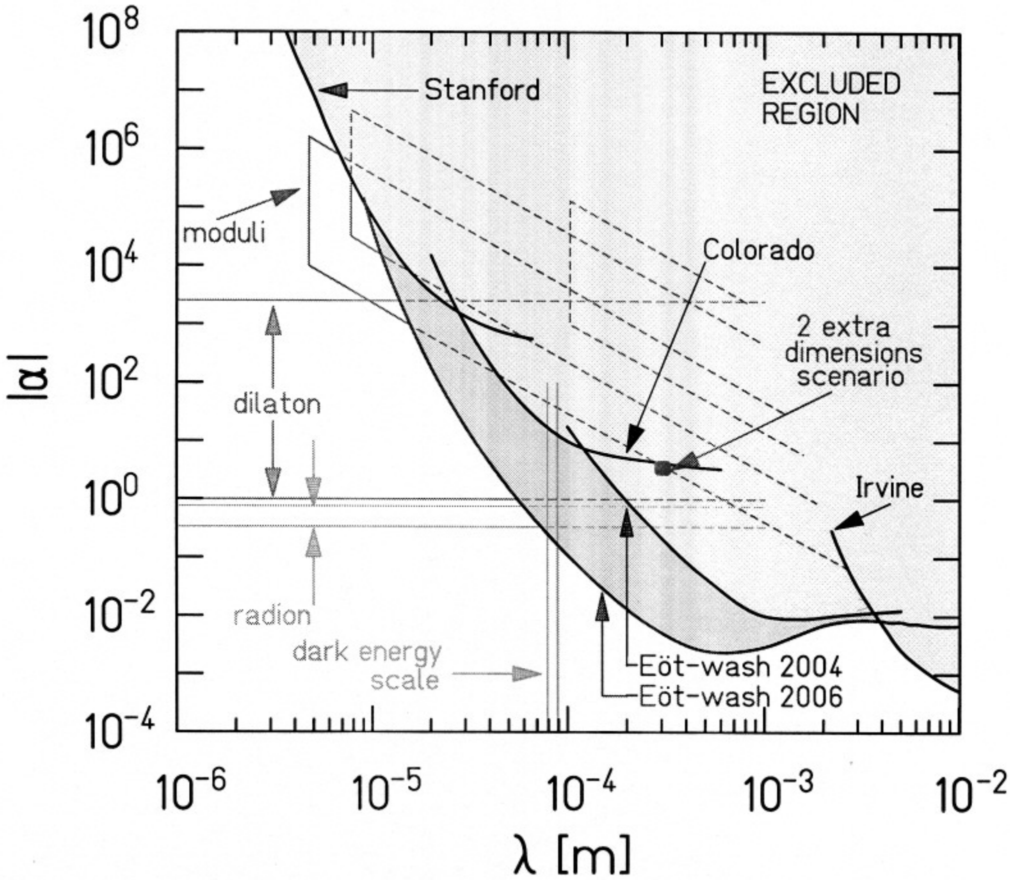
FIG. 1: Scale drawing of our detector and attractor. The 3 small spheres near the top of the detector were used for a continuous gravitational calibration of the torque scale. Four rectangular plane mirrors below the spheres are part of the twist-monitoring system. The detector's electrical shield is not shown.

mm thick molybdenum disc with 42 3.178-mm-diameter holes mounted atop a thicker tantalum disc containing 21 6.352-mm-diameter holes. The gravitational interaction between the missing masses of the detector and attractor holes applied a torque on the detector that oscillated 21 times for each revolution of the attractor, giving torques at 21ω, 42ω, 63ω, etc. that we measured by monitoring the pendulum twist with an autocollimator system. The holes in the lower attractor ring were displaced azimuthally by 360/42 degrees and were designed to nearly cancel the 21ω torque *if the inverse-square law holds.* On the other hand, an interaction that violated the inverse-square law, which we parameterize as a single Yukawa

$$V(r) = -G\frac{m_1 m_2}{r}[1 + \alpha \exp(-r/\lambda)] , \qquad (1)$$

would not be appreciably canceled if λ is less than the 1 mm thickness of the upper attractor disc. We minimized electromagnetic torques by coating the entire detector with gold and surrounding it by a gold-coated shield consisting of a tightly-stretched, 10 μm-thick, beryllium-copper membrane between the detector and attractor plus a copper housing that had small holes from:

Kapner et al, 2006



for assumed gravity potential

$$V(r) = -G \frac{m_1 m_2}{r} \left[1 + \alpha \exp\left(-r/\lambda\right) \right]$$

Assuming $M_* \sim 1\,\mathrm{TeV}$, which radius is required for n extra dimensions?

We have

$$\frac{1}{r} = M_* \left(\frac{M_*}{M_{Pl}}\right)^{\frac{2}{n}} = (1\,\mathrm{TeV}) \cdot 10^{-\frac{32}{n}}$$

for $M_* = 10^3\,\mathrm{GeV}$
$M_{Pl} = 10^{19}\,\mathrm{GeV}$

or

$$r \simeq 2 \cdot 10^{-19}\; 10^{\frac{32}{n}}\; m$$

For $\underset{\sim}{n=1}$: $r = 10^{13}\,m$ clearly excluded!

$\underset{\sim}{n=2}$: $r \simeq 2\,mm$ now excluded

<u>but</u>: $M_* \geq 3.2\,\mathrm{TeV}$ still possible!

$\underset{\sim}{n=3}$: $r \sim 10^{-8}\,m$ allowed

Note that in the ADD scenario the hierarchy problem is not really resolved but only translated into the problem of explaining why the extra dimensions are so large.

4) Brane and brane action

Branes have natural origin in string theory or as domain walls. In string theory, D-branes emerge as soliton solutions of equations of motion in supergravity approximation.

Here we will use low energy description: branes are assumed to be where they are without asking for their dynamical origin. At higher energies ($> \Lambda_{cut\text{-}off}$), however, dynamics of the brane will be important.

Consider 3-brane $\sim \mathbb{R}^4$ and additional dimensions X^n $(= \mathbb{R}^n, T^n, \dots)$

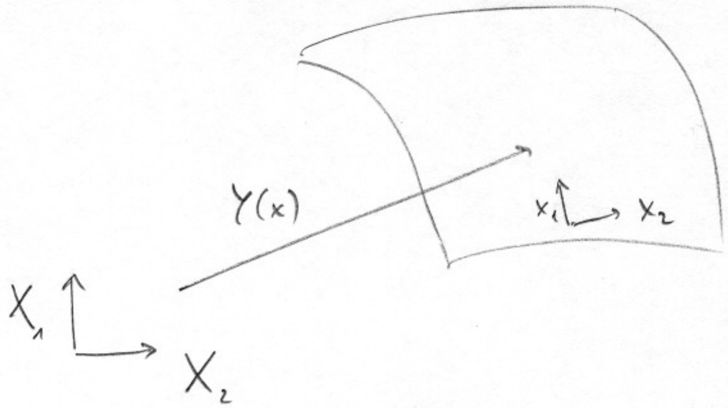
Coordinates in full space X^M, $M = 0, 1, \dots, 4+n$

" on the brane x_μ, $\mu = 0, 1, 2, 3$

" along extra dim. x_m, $m = 5, \dots, 4+n$

metric in higher dim. space $G_{MN}(X)$

Let position of the brane in higher-dim. space be $Y^M(x)$, and we have fields $\phi(x)$, $A_\mu(x)$, $\psi_L(x)$, ... on the brane.



Assuming flat brane in flat space as the vacuum,

$$G_{MN}(X) = \eta_{MN}$$
$$Y^M(x) = \delta^M_\mu x^\mu .$$

We have bulk ($\equiv$ total space) action

$$S_{bulk} = -\int d^{4+n}X \sqrt{|G|}\left(M_*^{n+2} R^{(4+n)} + \Lambda\right)$$

possible bulk cosmological constant

For effective action on the brane we need induced metric:

$$d^2s = G_{MN}\, dY^M(x)\, dY^N(x)$$
$$= G_{MN} \frac{\partial Y^M}{\partial x^\mu} dx^\mu \frac{\partial Y^N}{\partial x^\nu} dx^\nu$$

In flat case obtain $\eta_{\mu\nu}$ (flat 4d Mink. metric)

Now consider brane induced part of action.
Needs to be invariant under general coordinate transformations of bulk coord. X and under general coord. transf.s of brane coord. x. The latter corresponds to reparametrisations of brane (surface) which cannot have physical significance.
→ will get 4d Lorentz invariance on brane.
(Technically, this requires to contract all bulk indices among themselves, and similarly for brane coord. indices.)
Hence general form of brane action:

$$S_{brane} = \int d^4x \sqrt{|g|} \Big[-f^4 - R^{(4)} + \frac{g^{\mu\nu}}{2} D_\mu \phi D_\nu \phi - V(\phi) - \frac{1}{4} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} + \dots \Big]$$

f^4 is possible energy density on brane
= <u>brane tension</u>

We assume brane tension small to be able to neglect back-reaction on background.
(Will be different later in RS scenario.)

4d reparametrisation invariance $x \to x'(x)$ requires four additional gauge fixing conditions. Possible choice

$$Y^\mu(x) = x^\mu$$

$\to$ only $Y^m(x)$, $m = 5, \dots, 4+n$, correspond to physical degrees of freedom.

Consider fluctuations of bulk metric G_{MN} around flat space background:

$$G_{MN} = \eta_{MN} + \frac{1}{2 M_*^{n/2+1}} h_{MN} ,$$

such that graviton h_{MN} has dimension $\frac{n}{2}+1$, as is usual for boson in $4+n$ dimensions. Factor $\frac{1}{2}$ required to get canonically normalized kinetic term when expanding Einstein-Hilbert action in h.

Expanding leading term in brane action ($\leftarrow$ coord. Y^M of brane),

$$\int d^4x \sqrt{|g|} \left[-f^4 + \dots \right]$$

with $g_{\mu\nu} = G_{MN} \partial_\mu Y^M \partial_\nu Y^N = \eta_{\mu\nu} + \partial_\mu Y^m \partial_\nu Y_m$

and $\det g = -\partial_\mu Y^m \partial^\mu Y_m$

gives $S = \int d^4x \, f^4 \partial_\mu Y^m \partial^\mu Y_m$

Note that negative tension $T = f^4$ would give negative kinetic energy term for Y^m, hence a ghost - signalling instability of a negative tension brane configuration.
(However, projecting out negative kinetic energy parts is possible $\rightarrow$ orbifolds.)

5) Coupling of SM fields to bulk gravitons

How does matter (SM) on the brane interact with the various graviton modes?
$\rightarrow$ construct interaction Lagrangian
SM fields see only induced metric

$$g_{\mu\nu}(x) = G_{MN}(x)\, \partial_\mu Y^M \partial_\nu Y^N$$

Let us now concentrate on bulk graviton modes and set brane fluctuations to zero, that is $Y^\mu = \delta^\mu_\mu x^\mu$, $Y^m = 0$.
Then $g_{\mu\nu}(x) = G_{\mu\nu}(x_\mu, x^m = 0)$

Brane action is

$$S = \int d^4x \sqrt{|g|}\; \mathcal{L}_{SM}(g_{\mu\nu}, \phi, A, \psi, \dots)$$

Writing arguments explicitly,

$$S_{SM}[g,\phi,\dots] = \int d^4x \sqrt{|g|}\; \mathcal{L}_{SM}(g,\phi,\dots)$$

Expanding around flat induced metric:

$$S_{SM}[g,\phi,\dots] = \int d^4x\; \mathcal{L}_{SM}(\eta,\phi,\dots)$$

$$+ \int d^4x \left.\frac{\delta S_{SM}}{\delta g_{\mu\nu}(x)}\right|_{g=\eta} \delta g_{\mu\nu}(x) + \dots$$

with

$$\delta g_{\mu\nu}(x) = \frac{1}{M_*^{\frac{n}{2}+1}}\, h_{\mu\nu}(x)$$

Now

$$T^{\mu\nu}_{SM} = \frac{1}{\sqrt{|g|}} \left.\frac{\delta S_{SM}}{\delta g_{\mu\nu}(x)}\right|_{g=\eta}$$

is energy-momentum tensor of SM matter.

$\rightarrow$ interaction of SM matter to graviton is

$$S_{int} = \int d^4x\; T^{\mu\nu} \frac{h_{\mu\nu}(x)}{M_*^{\frac{n}{2}+1}}$$

Note linear coupling of graviton to energy-momentum tensor. (Actually, this can be viewed as definition of $T^{\mu\nu}$)

In above $h_{\mu\nu}(x)$ is superposition of KK modes. For toroidal compactification of extra dim. with volume $V_n = (2\pi R)^n$ we can write

$$h_{MN}(x,y) = \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \frac{h_{MN}^{\vec{k}}(x)}{\sqrt{V_n}} e^{i\vec{k}\cdot\vec{y}/R}$$

$y^n = x^n$ for extra dim.

Hence coupling of SM matter to individual KK modes:

$$\sum_{\vec{k}} \int d^4x \, T^{\mu\nu} \frac{1}{M_*^{\frac{n}{2}+1}} \frac{h_{\mu\nu}^{\vec{k}}}{\sqrt{V_n}} = \sum_{\vec{k}} \int d^4x \frac{1}{M_{Pl}} T^{\mu\nu} h_{\mu\nu}^{\vec{k}}$$

Thus coupling of each individual mode is of strength $\frac{1}{M_{Pl}}$. But since there are many of those modes, the total coupling to the bulk graviton is of strength $\frac{1}{M_*}$! (see above action before KK decomposition)

A closer look at different degrees of freedom of graviton:

graviton is $D \times D$ symmetric tensor, $D = n+4$, hence $\frac{1}{2} D(D+1)$ components.

Further, we can fix gauge corresponding to D-dim. general coord. invariance by harmonic gauge (for example)

$$\partial_M h^M_{\ N} = \frac{1}{2}\partial_N h^M_{\ M}$$

This is not yet complete gauge fixing, still gauge transformation

$$h_{MN} \to h_{MN} + \partial_M \epsilon_N + \partial_N \epsilon_M$$

with
$$\Box \epsilon_M = 0$$

left, hence another D conditions can be imposed. After that

$$\frac{1}{2}D(D+1) - 2D = \frac{1}{2}D(D-3)$$

degrees of freedom left.

In D = 4 that are 2 helicity states, D = 5 gives 5 components, D = 6 gives 9 etc.

→ higher-dim. graviton involves more fields!

Consider in the following the different components of graviton

⊕ 4D graviton and KK modes:

$$4\left\{\begin{pmatrix} \overbrace{G^{\vec{k}}_{\mu\nu}}^{4} & \\ & \end{pmatrix}\right.$$

$\vec{k}$ is n-comp. vector,

For no sources they satisfy

$$\left(\Box + \hat{k}^2\right) G_{\mu\nu}^{\vec{k}} = 0$$

with

$$\hat{k}^2 = \sum_{i=1}^{n} \left|\frac{k_i}{R}\right|^2 .$$

Above eq. implies 10 components, but 5 eliminated by gauge conditions,

$$\partial^\mu G_{\mu\nu}^{\vec{k}} = 0 \quad , \quad G_\mu^{\mu\,\vec{k}} = 0$$

→ <u>massive</u> graviton in 4D : 5 d.o.f (eats one massless vector + massless scalar)

⊕ 4D <u>vector</u> and KK modes

$$\left(\begin{array}{c|c} & V_{\mu j}^{\vec{k}} \\ \hline V_{j\mu}^{\vec{k}} & \end{array}\right)$$

$n-1$ such KK towers, since graviton has eaten one. This expressed in constraint

$$\hat{k}^j V_{\mu j}^{\vec{k}} = 0 .$$

In addition, usual Lorentz gauge

$$\partial^\mu V_{\mu j}^{\vec{k}} = 0 .$$

Massive vectors have eaten a scalar.

⊕ 4D scalars and KK modes

$$\left(\begin{array}{c|c} & \\ \hline & S^{\vec{k}}_{ij} \end{array}\right)$$

Originally, $\frac{1}{2} n(n+1)$. But one eaten by graviton, and $(n-1)$ eaten by vectors.

One scalar mode is special since its zero-mode sets the size of the internal manifold. This special scalar is called radion.

Hence $\frac{1}{2} n(n+1) - n - 1 = \frac{1}{2}(n^2 - n - 2)$ scalars left.

n eaten fields correspond to constraint

$$\hat{k}^j S^{\vec{k}}_{jk} = 0 ,$$

separating out radion by constraint

$$S^{\vec{k}j}_{j} = 0 .$$

Then radion given by $h^{\vec{k}j}_{j}$.

Total # of d.o.f.

$$5 \text{ (graviton)} + 3(n-1) \text{ (vectors)}$$
$$+ \frac{1}{2}(n^2 - n - 2) \text{ (scalars)} + 1 \text{ (radion)}$$
$$= \frac{1}{2}(4+n)(1+n) = \frac{1}{2} D (D-3)$$

as expected.

Explicitly (with usual normalization) in unitary gauge:

radion: $H^{\vec{k}} = \frac{1}{k} h_j^{\vec{k}j}$

Scalars: $S_{ij}^{\vec{k}} = h_{ij}^{\vec{k}} - \frac{k}{n-1}\left(\eta_{ij} + \frac{\hat{k}_i \hat{k}_j}{\hat{k}^2}\right) H^{\vec{k}}$

vectors: $V_{\mu j}^{\vec{k}} = \frac{i}{\sqrt{2}} h_{\mu j}^{\vec{k}}$

gravitons: $G_{\mu\nu}^{\vec{k}} = h_{\mu\nu}^{\vec{k}} + \frac{k}{3}\left(\eta_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{\hat{k}^2}\right) H^{\vec{k}}$.

where $\hat{k}_i = \frac{k_i}{R}$, $k = \sqrt{\frac{3(n-1)}{n+2}}$.

In presence of sources $T^{\mu\nu}$ the equation of motion for above fields is

$$(\Box + \hat{k}^2)\begin{pmatrix} G_{\mu\nu}^{\vec{k}} \\ V_{\mu j}^{\vec{k}} \\ S_{ij}^{\vec{k}} \\ H^{\vec{k}} \end{pmatrix} = \begin{pmatrix} \frac{1}{M_{Pl}}\left[-T_{\mu\nu} + \frac{1}{3}\left(\eta_{\mu\nu} + \frac{\partial_\mu\partial_\nu}{\hat{k}^2}\right) T_\mu^\mu\right] \\ 0 \\ 0 \\ \frac{k}{3M_{Pl}} T_\mu^\mu \end{pmatrix}$$

Hence radion is only field besides 4D graviton that couples to SM sources! Vector and other scalars are not important for SM matter - bulk graviton coupling.

As example, consider QED on the brane,

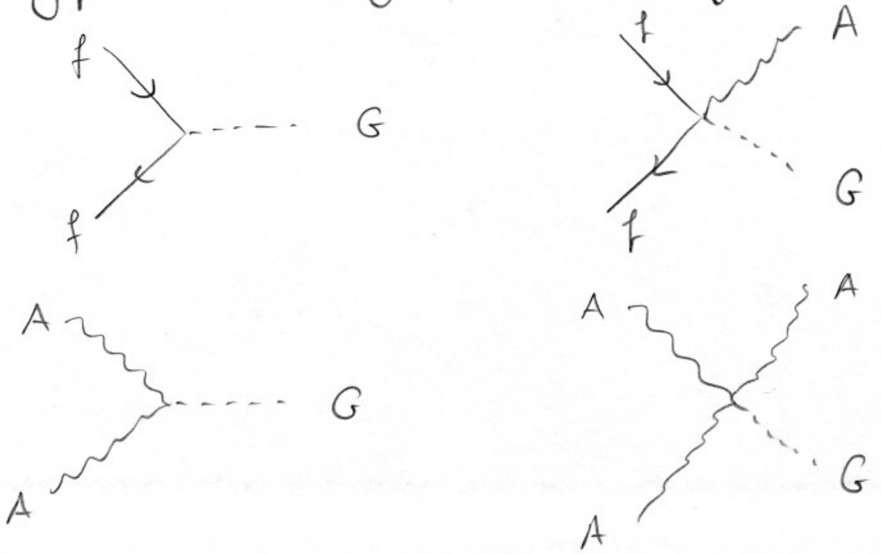
$$\mathcal{L} = \sqrt{|g|}\left(i\bar{\psi}\gamma^a D_a \psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}\right)$$

then energy-momentum tensor is

$$T_{\mu\nu} = \frac{i}{4}\bar{\psi}(\gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu)\psi - \frac{i}{4}(\partial_\mu \bar{\psi}\gamma_\nu + \partial_\nu \bar{\psi}\gamma_\mu)\psi$$
$$+ \frac{1}{2} eQ\, \bar{\psi}(\gamma_\mu A_\nu + \gamma_\nu A_\mu)$$
$$+ F_{\mu\lambda}F_\nu{}^\lambda + \frac{1}{4}\eta_{\mu\nu} F^{\lambda\rho}F_{\lambda\rho} .$$

→ Find QED - graviton couplings and Feynman rules from linear coupling to $h_{\mu\nu}$; similarly for whole SM.

Typical Feynman diagrams are



← (non-Abelian gauge theory)

and there are similar vertices for the radion.

6) Phenomenology of large extra dimensions

Recall that mass splitting of KK modes is extremely small:

$$\Delta m \sim \frac{1}{r} = M_* \left(\frac{M_*}{M_{Pl}}\right)^{\frac{2}{n}} = \left(\frac{M_*}{TeV}\right)^{\frac{n+2}{2}} 10^{\frac{12n-31}{n}} \, eV .$$

Hence a large number of KK modes is available for being produced in scattering processes at high energy. To deal with this large number we can turn the sum over KK modes into an integral (almost continuum due to small spacing). Let N be number of KK modes with extra-dim. momentum below k, then

$$dN = S_{n-1} \, k^{n-1} \, dk$$

with $S_n = (2\pi)^{n/2} \cdot \frac{1}{\Gamma(\frac{n}{2})}$ the surface of n-dim. sphere of unit radius.
Mass of a given KK mode is $m = \frac{|k|}{R}$, hence

$$dN = S_{n-1} \, m^{n-1} R^{n-1} \, dm \, R$$

$$= S_{n-1} \frac{M_{Pl}^2}{M_*^{n+2}} \, m^{n-1} \, dm$$

If $\frac{d\sigma_n}{dt}$ is differential cross section for production of an individual mode of mass m, then we get

$$\frac{d^2\sigma}{dt\,dm} = S_{n-1} \frac{M_{Pl}^2}{M_*^{n+2}} m^{n-1} \frac{d\sigma_n}{dt} .$$

Recall that coupling of SM fields to individual KK mode is $\sim \frac{1}{M_{Pl}}$, hence corresponding cross section $\sim \frac{1}{M_{Pl}^2}$. Therefore, inclusive cross section will be

$$\frac{d^2\sigma}{dt\,dm} \sim S_{n-1} \frac{m^{n-1}}{M_*^{n+2}} .$$

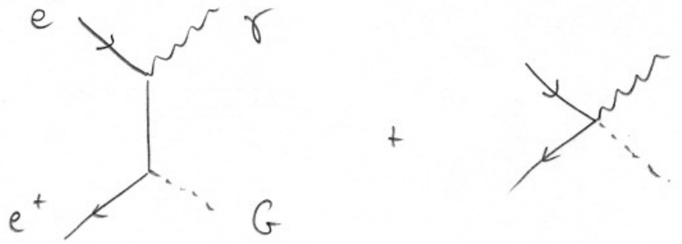
Here detailed phenomenology of many processes could start. We will only briefly discuss some interesting examples, concentrating on the results rather than the actual — sometimes lengthy — calculations.

* Graviton production at colliders

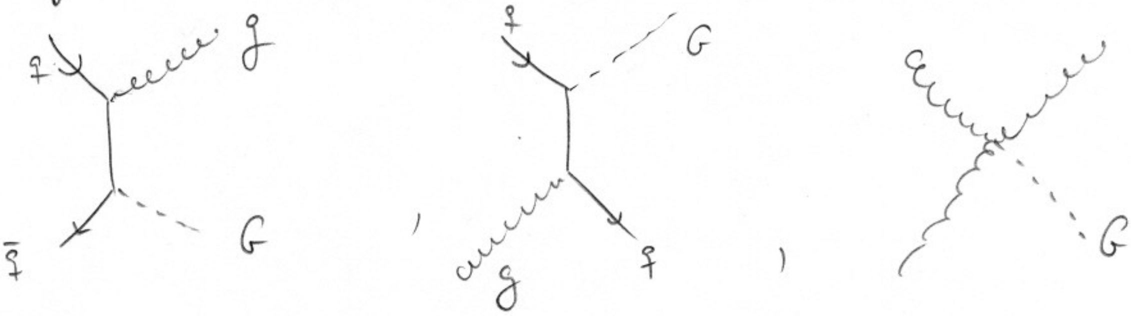
Typical process at e^+e^- - colliders is

$$e^+e^- \to \gamma G,$$

corresponding diagrams:



At hadron colliders one can have single - jet production from diagrams like



Individual graviton KK mode, once produced, has very long lifetime τ

$$\tau = \frac{1}{\Gamma} \sim \frac{M_{Pl}^2}{m^3} \quad \left(\text{since coupling to matter is} \sim \frac{1}{M_{Pl}}\right).$$

Therefore, graviton will not decay inside the detector, and will not be seen due to its weak interaction with matter.

$\to$ signature is missing energy!

In an e^+e^- collider a typical signal would be a single photon recoiling against missing energy, $e^+e^- \to \gamma + \not{E}_T$.
At hadron colliders one would see a single jet and missing energy.

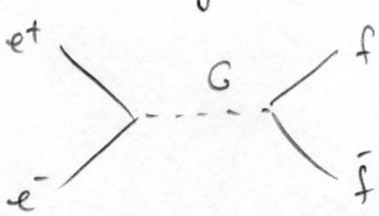
These processes have SM backgrounds (like Z production with initial state photon radiation and $Z \to$ neutrinos) which are rather small.

Importantly, such processes are often very different from SUSY processes.

As is clear from the general form of cross sections from higher-dim. perspective, such processes become relevant at energies $E \sim M_*$.

* virtual graviton exchange

At energies $E \geq M_*$ virtual graviton exchange becomes relevant:



with amplitude

$$A \sim \frac{1}{M_{pe}^2} \sum_{\vec{k}} \frac{1}{s - m_{kk}^2}$$

(Note: some problems due to UV divergence, $A \sim s^{(n-2)/2} \to$ UV completion?)

* Supernova cooling

Since graviton modes are weakly coupled they can carry away energy from supernovae. For temperature $T < r^{-1}$ no KK modes can be excited, but if $T \gg r^{-1}$ a large number $\sim (Tr)^n$ of modes can be produced. Each mode is coupled with strength $\frac{1}{M_{Pl}}$

$\rightarrow$ rate of graviton production is

$$\sim \frac{1}{M_{Pl}^2} (Tr)^n \sim \frac{T^n}{M_*^{n+2}}$$

with T typical temperature within a supernova ~ 30 MeV.

The normal cooling process is via neutrino emission. A considerable cooling via gravitons would hence affect the neutrino signal. No effect has been seen in SN 1987A, for example, where the signal was not shorter than expected.

This gives $M_* \gtrsim 30$ TeV for $n=2$.

* Supernova → neutron star transition

In a supernova many KK modes are produced near threshold, that is with low kinetic energy. Therefore they remain trapped in gravitational field of neutron star.

The decay time for $G \to \gamma\gamma$ is

$$\tau \sim 6 \cdot 10^9 \, yr \cdot \left(\frac{100 \, MeV}{m_G} \right)^3 .$$

Hence by now a significant fraction should have decayed into two hard photons, turning neutron stars into hard γ-sources. Non-observation of this implies

$$M_* \gtrsim 100 \, TeV \quad \text{for} \quad n=2 .$$

* Cooling into the bulk

At high temperatures in the early universe a considerable fraction of the energy on the brane can be carried away (into the bulk) via graviton production.

→ Compare cooling due to graviton production with ordinary cooling due to Hubble expansion.

Via expansion, the energy density changes as

$$\left.\frac{d\rho}{dt}\right|_{\text{expansion}} \sim -3H\rho$$

$$\sim -3\frac{T}{M_{Pl}^2}\rho$$

while graviton emission gives

$$\left.\frac{d\rho}{dt}\right|_{\text{grav-emission}} \sim \frac{T^n}{M_*^{n+2}}\rho$$

The two rates are equal at the so-called normalcy temperature T_*, below which normal expansion would dominate. It is given by

$$T_* \sim \left(\frac{M_*^{n+2}}{M_{Pl}}\right)^{\frac{1}{n+1}} = 10^{\frac{6n-9}{n+1}}\ \text{MeV}$$

After inflation, the reheating temperature should hence stay below the normalcy temperature to avoid production

of too much dark matter (gravitons in bulk) leading to overclosure.
For $n=2$, $T_* \sim 10\,\text{MeV}$, which is just barely sufficient for nucleosynthesis.
Generally, in these models baryogenesis is an enormously difficult problem due to the low temperature of the universe.

* Black hole production at colliders

Since the scale of quantum gravity is lowered to TeVs in the ADD model one can even expect black hole production at colliders.
Black holes are formed when the mass of an object is within the horizon corresponding to that mass.
The horizon for the mass of the Earth, for example, is $\sim 8\,\text{mm}$.
Most of the mass of the Earth is clearly outside the horizon, and no black hole is formed.

To calculate typical size of horizon in large extra dimensions, consider first 4d case. Schwarzschild solution is given by

$$ds^2 = \left(1 - \frac{GM}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{GM}{r}\right)} + r^2 d^2\Omega$$

and the horizon is the distance at which the coefficient of dt^2 vanishes,

$$r_H^{(4)} = GM$$

In $4+n$ dimensions the prefactor of dt^2 is replaced by

$$1 - \frac{GM}{r} \longrightarrow 1 - \frac{M}{M_*^{2+n} r^{1+n}} ,$$

hence the horizon size

$$r_H^{(4+n)} \sim \left(\frac{M}{M_*}\right)^{\frac{1}{1+n}} \frac{1}{M_*} .$$

The exact solution gives similar result with some numerical coefficient.

In a particle collision, a black hole would form if the collision takes place with sufficient energy ($\rightarrow$ mass) at impact parameter below $r_H^{(4+n)}$.

Hence a black hole of mass $M_{BH} = \sqrt{s}$ forms roughly with geometrical cross section,

$$\sigma \sim \pi r_H^2$$

$$\sim \frac{1}{M_*^2} \left(\frac{M_{BH}}{M_*} \right)^{\frac{2}{n+1}}$$

giving about $\frac{1}{\mathrm{TeV}^2} \sim 400$ pb.

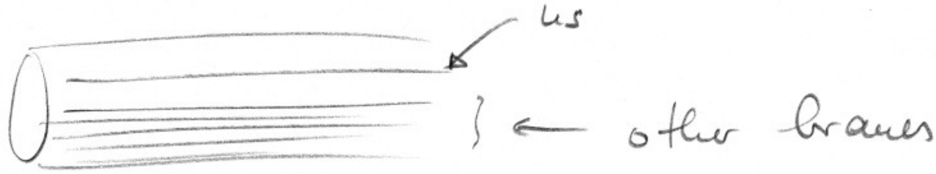
If so, the LHC would produce about 10^7 black holes per year!

They would be unstable and decay via Hawking radiation, in which all SM particles are produced with equal probability in a spherical distribution. A black hole of mass ~ 10 TeV would therefore decay into about ~ 50 particles with energies ~ 200 GeV each, leading to a spectacular signal.

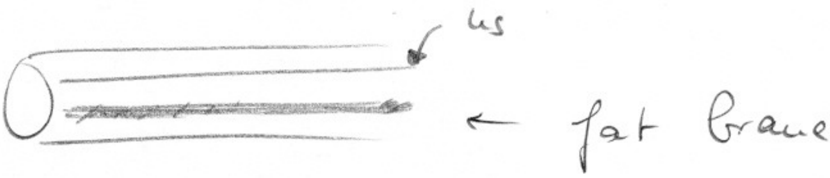
Various modifications of the ADD model have been proposed which can avoid some of the phenomenological bounds.

An example is to add further branes in the extra dimension.

- many other branes



- a "fat" brane



In such scenarios the gravitons would predominantly decay on the other branes, thus avoiding astrophysical bounds on production of hard photons via $G \to \gamma\gamma$.

4. Various models and mechanisms in flat extra dimensions

A number of different models with flat extra dimensions has been proposed, not all necessarily with large extra dimensions. Here we want to illustrate just a few interesting mechanisms.

1) Split fermions

If indeed extra dimensions are large and fundamental Planck scale is $M_* \sim 1\,\text{TeV}$, then the following issue emerges:

Quantum gravity generically breaks all global symmetries (but not gauge symmetries). Hence global symmetries can only be assumed to hold up to operators suppressed by the scale of quantum gravity.

Baryon number is an accidental global symmetry of the SM (every renormalizable operator consistent with gauge symmetry also conserves baryon number, without explicitly requiring it). But one can write non-renormalizable operators that violate baryon number.

If in the SM there is no new physics up to the GUT or Planck scale then those operators have a $\frac{1}{M_{GUT/Pl}}$ - suppression, leaving the proton sufficiently long-lived. However, new physics beyond the SM could induce proton decay. For example in the MSSM one has to introduce R-parity to avoid proton decay, otherwise suppression would be only $\sim \frac{1}{M_{SUSY}}$.

In large extra dimensions the situation is worse: quantum gravity is expected to be sizeable at Planck scale $M_* \sim$ TeV. Thus proton decay via an operator with three quarks and one lepton, for example, would have a small suppression

$$\frac{1}{M_*^2} \, QQQL$$

and would cause proton decay.

A nice solution to this problem in extra dimensions was proposed by Arkani-Hamed and Schmaltz. They make use of the extra dimension to suppress the dangerous operators. The idea is to localize the SM fermions at slightly different points along the extra dimension

→ "Split fermion scenario"

If the fermions have narrow wave functions along the extra dim. the dangerous operators get suppressed by the small overlap of the corresponding wave functions. This idea can also be used to generate the fermion mass hierarchy.

So far we have not discussed the issue of localizing fields. The following discussion shows how a "brane-like" object can emerge in field theory as a domain wall.

We consider a single extra dimension and try to localize fermions. For that we first need to know how to describe fermions in 5d. A fermion is a representation of the Poincaré group, and we need a representation of the

5d Clifford algebra

$$\{\Gamma_M, \Gamma_N\} = 2\eta_{\mu\nu}$$

to describe it. Such a representation is straightforward to find in terms of the well-known (from 4d) Dirac gamma-matrices:

$$\Gamma_\mu = \gamma_\mu^{(4d)} \quad , \quad \Gamma_5 = -i\gamma_5^{(4d)}$$

(In the following we will in 5d use again small γ's: $\gamma_M \equiv \Gamma_M$)

Note: in 5d this is the smallest representation of the Clifford algebra. Recall that in 4d in addition to the corresponding 4-component spinors one had also a 2dim representation given by Weyl fermions (chiral fermions)

Hence in 5d fermions are described by 4-component spinors. But now γ_5 is part of the algebra!

→ There are no chiral fermions like we had in 4d.

(Orbifold boundary conditions can also resolve this, see below.)

The 5d Lorentz invariants one can construct out of two four-component 5d spinors ψ_1 and ψ_2 are

$\bar{\psi}_1 \psi_2$ (like 4d Dirac mass term)

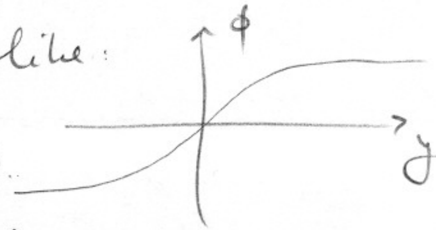
$\psi_1^T C_5 \psi_2$ (like 4d Majorana mass term)

where

$C_5 = \gamma^0 \gamma^2 \gamma^5$ is 5d charge conjugation

Now introduce a scalar field ϕ that forms a domain wall, like:

(note sign change at $y=0$)



This will lead to a physical example of a "fat brane", a brane of finite width.

Consider fermion in background $\phi(y)$ and assume Yukawa coupling to ϕ

$\rightarrow$ $S = \int d^4x \, dy \; \bar{\psi} \left[i\gamma_\mu \partial^\mu + i\gamma_5 \partial_y + \phi(y) \right] \psi$,

leading to 5d Dirac equation

$$\left[i\gamma_\mu \partial^\mu + i\gamma_5 \partial_y + \phi(y) \right] \psi = 0$$

We look for solutions which are <u>4d</u> left- or right handed modes, hence we expand

$$\psi(x,y) = \sum_n \langle y | L,n \rangle \, P_L \, \psi_n^{(4)}(x) + \sum_n \langle y | R,n \rangle \, P_R \, \psi_n^{(4)}(x)$$

with

$$P_{L,R} = \frac{1}{2} \left(1 \pm \gamma_5 \right)$$

To do so we diagonalize the y-dependent part of Dirac operator, hence we have to find solutions to (as usual squaring the operator)

$$\underbrace{\left[-\partial_y^2 + \phi(y)^2 \pm \dot{\phi}(y) \right]}_{\substack{= a a^+ \\ (a^+ a)}} \left| {L \atop R}, n \right\rangle = \mu_n^2 \left| {L \atop R}, n \right\rangle$$

$$\dot{\phi} = \frac{d}{dy} \phi$$

for $a = \partial_y + \phi(y)$, $a^+ = -\partial_y + \phi(y)$

With a, a^+ the problem can be formulated as a typical SUSY quantum mechanics problem:

$$Q = a\gamma^0 P_L$$
$$Q^+ = a^+\gamma^0 P_R$$

gives for the Hamiltonian

$$H = \{Q, Q^+\} \, .$$

For such problems it is known that the eigenvalues (here for L and R modes) always come in pairs, <u>except</u> possibly for zero modes.

→ zero modes could give chiral 4d fermions.

Therefore we consider only zero modes, $n=0$, they have $\mu_n^2 = 0$.

$H\psi = 0$ is solved by $Q\psi = 0$, $Q^+\psi = 0$

→ $$\left[\pm \partial_y + \phi(y) \right] \left| {}^{L}_{R}, 0 \right\rangle = 0 \, ,$$

with solutions

$$\langle y | L, 0 \rangle \sim \exp\left[- \int_0^y \phi(s)\, ds \right]$$
$$\langle y | R, 0 \rangle \sim \exp\left[\int_0^y \phi(s)\, ds \right]$$

In infinitely large extra dim. they cannot be both normalizable.

→ Only one of them is relevant and we obtain a chiral zero-mode.

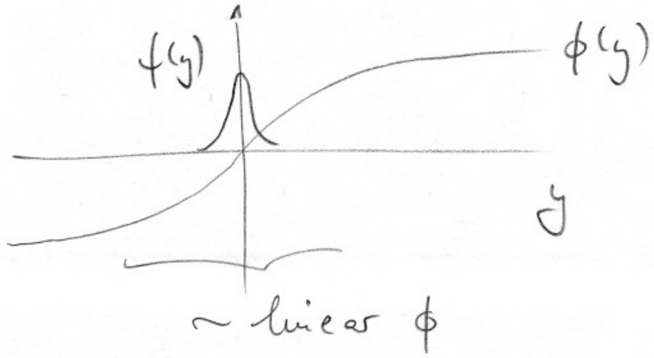
[In "realistic" models with a finite extra dimension the other mode can enter as well, but would be localized "at the other end of the extra dim."]

Assuming for example a linear domain wall (linear at least in a sufficiently large region around $y=0$)

$$\phi(y) \sim 2\mu^2 y$$

we have

$$\langle y | L, 0 \rangle = \frac{\sqrt{\mu}}{\left(\frac{\pi}{2}\right)^{1/4}} e^{-\mu^2 y^2}$$



while $\langle y | R, 0 \rangle$ is not normalizable and decouples.

Hence we obtain one left-handed fermion plus an infinite tower of massive Dirac fermions.

The zero-mode has a Gaussian wave function centered around $y=0$, that is where $\phi(y)$ changes sign. $\rightarrow$ The chiral fermion is dynamically localized to the domain wall. (often the name "domain wall fermion" is used, they are extensively used in lattice gauge theory.)

Next we put several fermions into the bulk. Then the action is

$$S = \int d^4x\, dy\; \bar{\Psi}_i \left[i\gamma_\mu \partial^\mu + i\gamma_5 \partial_y + \lambda \phi(y) - m \right]_{ij} \Psi_j$$

with a general mass matrix m_{ij} and a matrix λ_{ij} of Yukawa couplings.
For simplicity we assume $\lambda_{ij} = \delta_{ij}$ so that we can diagonalize m_{ij} with eigenvalues m_i.
We can find solutions in an analogous way, but now chiral fermions are localized around the zeros of

$$\phi - m_i \,.$$

If the domain wall is linear in a sufficiently large region, this gives fermions localized at

$$y^i = \frac{m_i}{2\mu^2},$$

that is, the m_i fix the relative position of the fermions along y.

Let us now turn to couplings in such models. For the SM we need fields of type:

doublet lefthanded
singlet righthanded,

however, only left handed fields emerge.
→ take charge conjugation for right-handed fields
Hence consider doublet L, singlet $\bar{E}^c$.
Consider Yukawa Couplings to bulk Higgs field connecting fermions:
5d action is

$$S = \int d^5x \; \bar{L} \left(i\gamma^M \partial_M + \phi(y) \right) L$$
$$+ \int d^5x \; \bar{E}^c \left[i\gamma^M \partial_M + \phi(y) - m \right] E^c$$
$$+ \int d^5x \; \kappa \; H L^T C_5 E^c$$

As before we find that the lefthanded fermion l from L is localized at $y=0$, while the righthanded e^c from E^c is localized at $y = r = \frac{m}{2\mu^2}$.

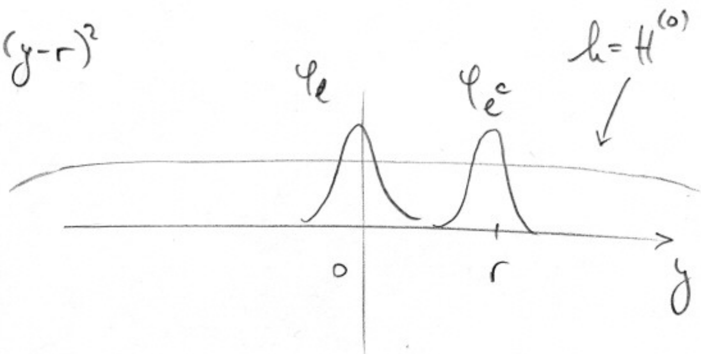
Then the <u>effective 4d Yukawa coupling</u> for zero modes will be

$$\int d^4x \, \kappa H \, l e^c \int dy \; \varphi_l(y) \, \varphi_{e^c}(y)$$

zero mode wavefunctions

but

$$\int dy \; \varphi_l(y) \, \varphi_{e^c}(y)$$
$$= \frac{\sqrt{2}\mu}{\sqrt{\pi}} \int dy \; e^{-\mu^2 y^2} e^{-\mu^2 (y-r)^2}$$
$$= e^{-\mu^2 r^2 / 2}$$



→ effective 4d Yukawa coupling is

$$\kappa \, e^{-\mu^2 r^2 / 2}$$

which is <u>exponentially small</u> even for bulk Yukawa coupling of order $O(1)$.

→ Split fermions can generate a large fermion hierarchy by relatively small splittings in the extra dimensions.

In this scenario fermion masses are thus related to geography in the extra dimension!

Similarly, one can suppress proton decay by localizing quarks and leptons at different ends of the fat brane. A dangerous operator would be

$$\int dy \, \frac{1}{M_*^3} \left(Q^T C_5 L \right) \left(U^{cT} C_5 D^c \right)$$

(as can be generated by quantum gravity). With overlap of wave functions one get an effective 4d coupling suppressed by

$$\int dy \left(e^{-\mu^2 y^2} \right)^3 e^{-\mu^2 (y-r)^2} \sim e^{-\frac{3}{4}\mu^2 r^2} .$$

→ An arbitrarily large suppression is possible without invoking any symmetry.

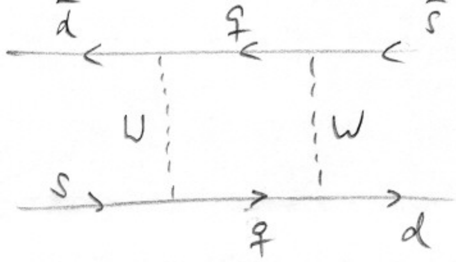
One can in fact construct large ($\sim \mathrm{TeV}^{-1} \cdot 10^{\frac{32}{n}}$) extra dimension scenarios with a fat brane that are "realistic": gauge and Higgs fields propagate throughout the fat brane, and only gravity propagates in full 5d space time. For a width of the fat brane $\sim \mathrm{TeV}^{-1}$ the KK modes of the gauge fields are sufficiently heavy. The typical width of the fermions in y is $\sim 0.1 - 0.01 \ \mathrm{TeV}^{-1}$.

2) Mediation of SUSY breaking via extra dimensions

Let us assume that the SM hierarchy problem is resolved by supersymmetry. Then the important question is: how is SUSY broken? Usually, in the MSSM (minimal supersymmetric SM) soft SUSY-breaking parameters are put in by hand, but they essentially only parametrize our ignorance of the real mechanism.

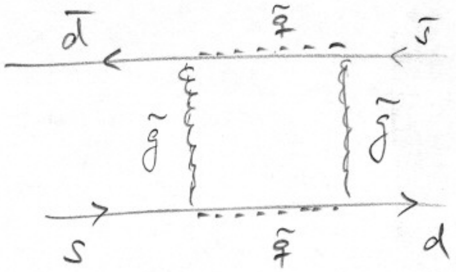
A very strong constraint on such parameters comes from the absence of flavor-changing neutral currents.

To illustrate the problem, consider K-$\bar{K}$ mixing which is very small in the SM due to the GIM mechanism:



This is prop. to off-diagonal elements of $V_{CKM}^{+} V_{CKM}$
$\rightarrow$ zero due to unitarity of V_{CKM}.

In the MSSM a possible contribution is



with squarks and gluinos, prop. to $V^+_{CKM} M^2_{squarks} V_{CKM}$

This is small only if $M_{squark} \sim$ unit matrix.

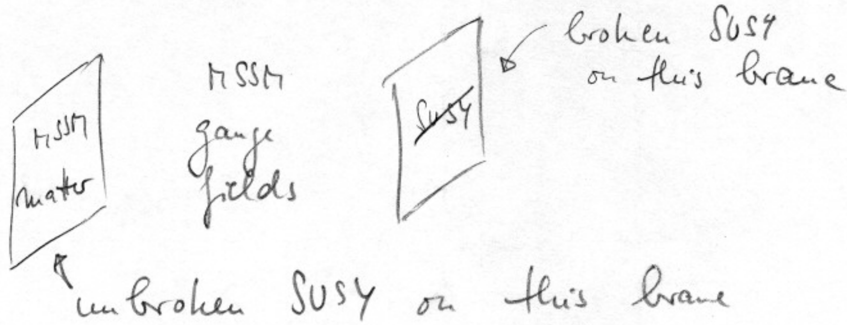
$\rightarrow$ "SUSY flavor problem": Why should squarks be degenerate in mass?

In extra dimensions, there are new ways of attacking the SUSY breaking problem. Two important proposals are

- anomaly mediation
- gaugino mediation

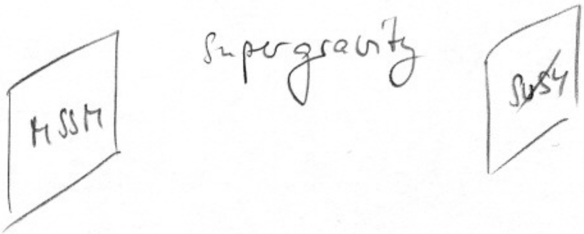
In both cases the assumption is that there are two branes separated by some (not necessarily large!) distance in an extra dimension. On one brane ("our world") SUSY is unbroken while on the other brane SUSY is explicitly broken.

In the case of gaugino mediated SUSY breaking one lets propagate the MSSM gauge fields in the bulk,



Then gauginos couple directly to SUSY breaking on the other brane, hence the leading breaking effect will be in the gaugino sector, while scalars experience SUSY breaking only via loop effects (with gauge fields propagating to the other brane) such that SUSY breaking effects are smaller in that sector.

In <u>anomaly mediated</u> SUSY breaking one has two similar branes but now lets only supergravity fields propagate in the bulk.



Now <u>all</u> SUSY-breaking effects in the MSSM are through loop effects. (Hence the name: this is a pure quantum effect, like anomalies in gauge theories)

3) Universal extra dimensions (UED)

(Appelquist, Cheng, Dobrescu)

In this model all SM fields propagate in all dimensions (hence the name "universal"). One assumes flat extra dimensions with compactification radius $R \sim \text{TeV}^{-1}$ (therefore one also speaks of "TeV-sized extra dim."), that is much smaller than in the ADD model. Essentially, the idea of this model is closer to the original KK model than to ADD.

Models of this kind have an interesting phenomenology, in particular they provide good candidates for dark matter, namely the lightest KK-state, which can be stable.

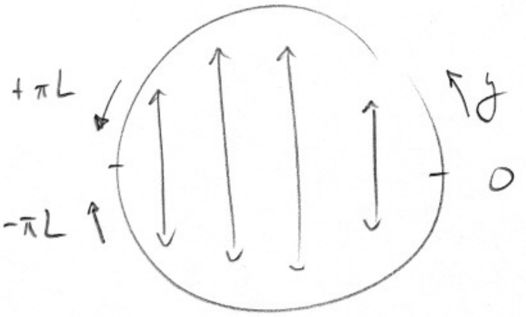
UED models are also often used as prototype models for effects of extra dimensions at colliders.

In these lectures, however, we will not consider any details of UED models.

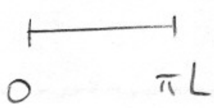
4) Orbifold compactification

As a motivation recall the chirality problem (no chiral representation of the Clifford algebra in 5d!). One way to resolve this is orbifold compactification, but orbifolds have also other effects that are interesting for model building.

An orbifold is in the simplest case obtained by compactifying one dimension on a circle S^1 with periodic boundary conditions and identifying y with $-y$:



The resulting physical space is an interval



with certain boundary conditions,
$\rightarrow$ the orbifold $S^1/\mathbb{Z}_2$, since $y \leftrightarrow -y$
corresponds to a (discrete) $\mathbb{Z}_2$-transformation.

To implement the identification $y \leftrightarrow -y$ on the fields, assign a parity $\mathcal{P}$ under the $\mathbb{Z}_2$-transformation to all fields (to be respected by the action).

For example:

$$\left.\begin{aligned} \mathcal{P} A_\mu &= + A_\mu \\ \mathcal{P} \psi_L &= + \psi_L \\ \mathcal{P} \psi_R &= - \psi_R \end{aligned}\right\} \quad \text{assuming vanishing 5d fermion mass}$$

Then all fields have

$$\phi(x^\mu, -y) = \mathcal{P}(\phi)\,(x^\mu, y),$$

and we also have periodicity

$$\phi(x^\mu, y + 2\pi L) = \phi(x^\mu, y).$$

Together these specify "orbifold boundary conditions" on the interval.

To satisfy these boundary conditions:

$$A_\mu^{(n)},\ \psi_L^{(n)} \sim \cos \frac{ny}{L}$$

$$\psi_R^{(n)} \sim \sin \frac{ny}{L}$$

Obviously, for $n=0$ $\psi_R^{(0)} = 0$, and there is no right-handed fermion zero-mode! Only a massless left-handed fermion is left.

→ Orbifold compactification can resolve the chirality problem.

Orbifolds can also be used in many other ways to project out unwanted states.

→ many applications of orbifolds in extra-dimensional model building.

5. Warped extra dimensions (Randall - Sundrum models)

So far we have considered flat extra dimensions and have neglected the brane tension.

→ What is actually the interplay of brane tensions and bulk cosmological constant?

As we will see, the mutual back-reaction of branes and bulk can be quite relevant: the effect of brane tension can be balanced by a bulk cosmological constant.

→ curved space with flat brane(s) becomes possible!

1) RS 1 model

In the RS 1 (Randall-Sundrum - 1) model consider one extra dimension with orbifold compactification $S^1/\mathbb{Z}_2$ and relevant interval $[0, L]$

→ Space is $\underbrace{\mathbb{R}^4}_{\text{4d Minkowski}} \times S^1/\mathbb{Z}_2$

The bulk action is

$$S_{bulk} = \int d^4x \int_{-L}^{L} dy \sqrt{|G|} \left(2M_*^3 R - \Lambda\right)$$

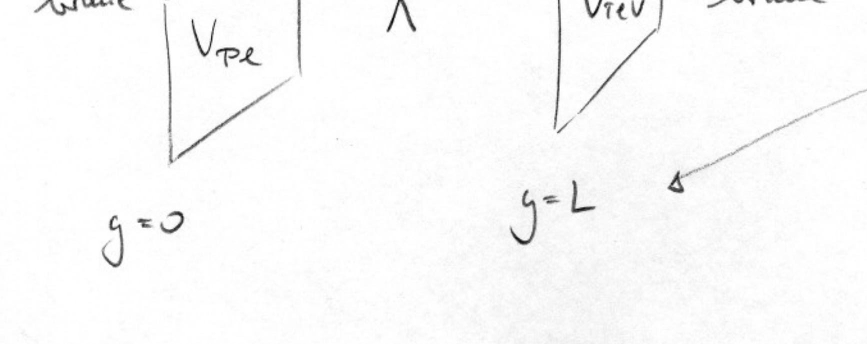
$$= 2 \int d^4x \int_0^L dy \sqrt{|G|} \left(2M_*^3 R - \Lambda\right)$$

to see how orbifold boundary conditions act it is convenient to study two copies glued together back-to-back

Then put two branes – called Planck-brane and TeV-brane – at $y=0$ and L that have tensions V_{PE}, V_{TeV} (4d cosmological constants) and matter fields described by $\mathcal{L}_{PE}$ and $\mathcal{L}_{TeV}$. Both couple to the 4d components of the bulk metric, that is to induced metric

$$g_{\mu\nu}^{PE}(x^\mu) = G_{\mu\nu}(x^\mu, y=0)$$

$$g_{\mu\nu}^{TeV}(x^\mu) = G_{\mu\nu}(x^\mu, y=L)$$



Note: L is not necessarily large

Hence the brane actions

$$S_{Pl} = \int d^4x \sqrt{|g_{Pl}|}\,\left(\mathcal{L}_{Pl} - V_{Pl}\right)$$

$$S_{TeV} = \int d^4x \sqrt{|g_{TeV}|}\,\left(\mathcal{L}_{TeV} - V_{TeV}\right)$$

→ total action:

$$S = S_{bulk} + S_{Pl} + S_{TeV}$$

Now consider only effect of gravity and tension / cosmological constant, and neglect matter on the branes.

→ 5d Einstein equations for this:

$$\sqrt{|G|}\left(R_{MN} - \frac{1}{2} G_{MN} R\right) =$$
$$= -\frac{1}{4M_*^3}\Big[\Lambda \sqrt{|G|}\, G_{MN}$$
$$+ V_{TeV}\sqrt{|g_{TeV}|}\, g^{TeV}_{\mu\nu}\, \delta^\mu_M \delta^\nu_N\, \delta(y-L)$$
$$+ V_{Pl}\sqrt{|g_{Pl}|}\, g^{Pl}_{\mu\nu}\, \delta^\mu_M \delta^\nu_N\, \delta(y)\Big]$$

We look for a solution that respects 4d Poincaré invariance in x^μ-directions

→ non-factorizable ansatz for 5d metric:

$$ds^2 = e^{-2\sigma(y)}\, \eta_{\mu\nu}\, dx^\mu dx^\nu - dy^2$$

With this ansatz Einstein eq. gives

$$6\sigma'^2 = -\frac{\Lambda}{4M_*^3} \qquad (*)$$

$$3\sigma'' = \frac{1}{4M_*^3} V_{Pl}\,\delta(y) + \frac{1}{4M_*^3} V_{TeV}\,\delta(y-L) \qquad (**)$$

For orbifold we require symmetry w.r.t. $y \leftrightarrow -y$, so from $(*)$:

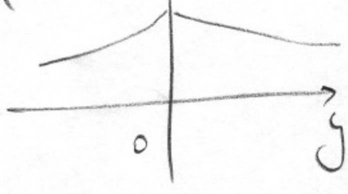
$$\sigma = |y| \sqrt{\frac{-\Lambda}{24 M_*^3}}$$

and a reasonable (real-valued) solution requires $\Lambda < 0$, that is a constant <u>negative curvature</u>. Such a space is called an <u>Anti-de Sitter space</u>, in 5d also denoted by <u>AdS_5</u>.

More precisely, since we are dealing with a space bounded by two branes, a <u>slice of AdS_5</u>.

For second derivative of σ recall orbifold boundary conditions

→ a cusp is to be expected and hence the δ-fct. pieces in eq. $(**)$ for σ''.



With the solution to (*) we hence have to fulfill in addition for $-L \leq y \leq L$:

$$\sigma'' = 2\sqrt{\frac{-\Lambda}{24 M_*^3}} \left[\delta(y) - \delta(y-L) \right] .$$

A solution is obtained for (**) only if V_{TeV}, V_{Pl} and Λ are related. Then they can be expressed in terms of a single scale k:

$$V_{Pl} = - V_{TeV} = 24 M_*^3 k$$
$$\Lambda = - 24 M_*^3 k$$

Obviously, a fine-tuning is required to have a solution.

Note also that L is not determined and remains as a free parameter – a "modulus". Changes of L correspond to a massless field, the radion.

To make the model realistic, some stabilization mechanism is required (see below).

In summary, we have

$$\sigma(y) = k|y|$$

with

$$k = \sqrt{\frac{-\Lambda}{24 M_*^3}}$$

k is called the <u>AdS curvature</u>, while $l = \frac{1}{k}$ is the <u>AdS length</u>.

The bulk metric is

$$ds^2 = \underbrace{e^{-2k|y|}}_{\text{warp factor}} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2$$

Recall that the TeV-brane has negative tension ($\rightarrow$ "negative tension brane" or "weak brane") and the Planck-brane has positive tension ($\rightarrow$ "positive tension brane", "gravity brane") in order to have a solution to the Einstein equation.

2) Physical scales in RS1

Now consider physical mass scales in this model.

We assume that matter resides on the TeV-brane, for example the Higgs field. The action for the latter is

$$S^{Higgs} = \int d^4x \sqrt{|g^{TeV}|} \left[g^{TeV}_{\mu\nu} D^\mu H D^\nu H - V(H) \right]$$

with the Higgs potential

$$V(H) = \lambda \left(H^+ H - v^2 \right)^2 .$$

With the warp factor the induced metric is

$$g^{TeV}_{\mu\nu} = e^{-2kL} \eta_{\mu\nu} .$$

Inserting into the action:

$$S^{Higgs} = \int d^4x \; e^{-4kL} \left[e^{2kL} \eta_{\mu\nu} D^\mu H D^\nu H + \lambda \left(H^+ H - v^2 \right)^2 \right]$$

To obtain canonically normalized Higgs field we re-define $\hat{H} = e^{-kL} H$, so

$$S^{Higgs} = \int d^4x \left\{ \eta_{\mu\nu} D^\mu \hat{H} D^\nu \hat{H} - \lambda \left[\hat{H}^+ \hat{H} - \left(e^{-kL} v \right)^2 \right]^2 \right\}$$

That is the standard form of the Higgs action, but with a VEV that is warped down to

$$\tilde{v}_{Higgs} = e^{-kL} v .$$

Note that this scale (which sets all SM mass parameters!) is now <u>exponentially smaller</u> than usually.

One finds that this mechanism is completely general: <u>all</u> masses are exponentially suppressed on the TeV-brane. But this is not the case on the Planck-brane (positive tension brane) since there the warp factor equals 1.

That phenomenon is sometimes called "red shifting" of all energy scales away from the positive tension brane.

Interpretation: all fundamental mass parameters are of $\mathcal{O}(M_{Pl})$, including $v \sim 0.1\, M_{Pl}$ on the Planck brane, and are warped down exponentially on the TeV brane.

Then to obtain

$$\tilde{v} \sim 0.1 \cdot e^{-kL} M_{Pl} \sim 0.1\ \text{TeV}$$

requires only

$$k \cdot L \sim 35 .$$

That is a rather simple physical requirement without too much fine-tuning. Then all dimensionful parameters on the TeV-brane are warped down, and for that one only needs that the size of the AdS space L is parametrically larger than the inverse curvature.

Hence the actual parameter of the extra dimension in RS 1 is <u>not</u> the size L, but rather the (inverse) curvature of the AdS space.

[Another possible interpretation is that all fundamental parameters are of $\mathcal{O}(\text{TeV})$ and M_{Pl} is large due to "warping up" in the opposite direction.]

What is 4d effective scale of gravity (4d Planck scale)?

$\to$ find coefficient of $R^{(4)}$ upon integrating out y-direction.

The 4d graviton is embedded as (see later)

$$ds^2 = e^{-2k|y|}\left[\eta_{\mu\nu} + h_{\mu\nu}(x)\right]dx^\mu dx^\nu - dy^2$$

Then $R^{(5)}_{\mu\nu}$ contains $R^{(4)}_{\mu\nu}$ calculated from $h_{\mu\nu}$ (since $R_{\mu\nu}$ is invariant under rescaling of the metric)

$$\to \quad S = -M_*^3 \int d^5x \sqrt{G}\, R^{(5)}$$

contains

$$-M_*^3 \int d^5x\, e^{-4k|y|} \sqrt{g^{(4)}}\, e^{2k|y|}\, R^{(4)}$$

Hence

$$M_{Pl}^2 = M_* \int_{-L}^{L} e^{-2k|y|}\, dy$$

$$= \frac{M_*^3}{k}\left(1 - \underbrace{e^{-2kL}}_{\text{small}}\right)$$

[note: this result is finite for $L \to \infty$!?]

Then it is natural that

$$M_{Pl} \sim k \sim M_*$$

are all of Planck scale size.

M_{Pl} barely depends on size L of the extra dimension! That is drastically different from the case of flat extra dim.!

Thus M_{Pl} in 4d remains large $\sim M_*$, and therefore gravity is weak!
→ The 4d hierarchy problem is resolved in this model!

3) Radius stabilization in RS1
(Goldberger - Wise mechanism)

Problem is that in RS1 as discussed so far there is no potential for the size-parameter L, implying the existence of a massless scalar field (radion).
→ Why is the size $k \cdot L \sim 35$ natural, and how could it be stable?

In the Goldberger - Wise mechanism this problem is fixed by introducing a massive scalar field in the bulk in a suitable way.

To obtain a potential for L with a minimum one needs two "forces" pulling in opposite directions (like for stable orbits in a two-body problem). Here the mass of the scalar field will tend to make the size smaller. In addition, one introduces a nontrivial profile of the scalar field VEV along the y-direction which tends to make the size larger (in order to have a smaller slope).

Technically, the latter is done by introducing <u>brane potentials</u> which fix the VEV at 0 and L at two different values.

The corresponding actions are

$$S^{\phi}_{bulk} = \frac{1}{2} \int d^4x \int_{-L}^{L} \sqrt{G} \left(G^{AB} \partial_A \phi \, \partial_B \phi - m^2 \phi^2 \right)$$

$$S^{\phi}_{Pl} = - \int d^4x \sqrt{g_{Pl}} \, \lambda_{Pl} \left(\phi^2 - v_{Pl}^2 \right)^2$$

$$S^{\phi}_{TeV} = - \int d^4x \sqrt{g_{TeV}} \, \lambda_{TeV} \left(\phi^2 - v_{TeV}^2 \right)^2$$

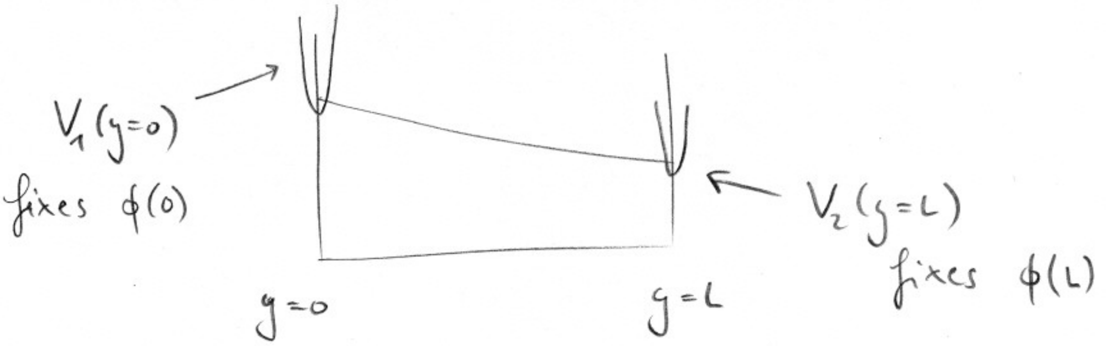
The general solution in the bulk is found as

$$\phi(y) = e^{2k|y|}\left(A\, e^{\nu k|y|} + B\, e^{-\nu k|y|}\right)$$

with

$$\nu = \sqrt{4 + \frac{m^2}{k^2}}$$

The brane potentials are chosen such that the situation looks like



Then one can write a potential for L, $V(L)$, and minimize it. One finds for the minimum

$$k \cdot L = \frac{4}{\pi}\, \frac{k^2}{m^2}\, \ln\left(\frac{v_{Pl}}{v_{TeV}}\right)$$

For $k \cdot L \simeq 35$ one needs $k > m$, but not by much, which is a simple requirement without fine-tuning.

Note that due to the potential for L (that is due to stabilization) the radion acquires a mass. → important for radion phenomenology!

The Goldberger - Wise mechanism was later improved by including the backreaction of the ϕ - energy density on the space - time curvature.

4) Gravity in RS1

Now study KK - decomposition of graviton in AdS background.

Usual expectation from flat extra dim. on S^1 is:

zero modes: $\left\{ \begin{array}{ll} \text{graviton} & \\ \text{vector} & (\text{"graviphoton"}) \\ \text{scalar} & (\text{"graviscalar"}) \end{array} \right.$

+ massive graviton at higher KK levels (having eaten scalar + vector)

But here situation is different due to orbifold! Consider generic form of metric with fluctuations:

$$ds^2 = e^{-2k|y|} g_{\mu\nu}\, dx^\mu dx^\nu + A_\mu\, dx^\mu dy - b^2 dy^2$$

contains graviton (see below) — vector fluctuation — scalar fluctuation

Since ds^2 is symmetric under $y \leftrightarrow -y$, A_μ has to change sign under that transf. and hence cannot have a zero-mode. Therefore there will only be a scalar and the graviton zero-mode, plus massive gravitons at higher KK levels.

We first concentrate only on the graviton and set b to zero.

To find KK expansion of the graviton go to <u>conformal frame</u>,

$$ds^2 = e^{-A(z)} \left[(\eta_{\mu\nu} + h_{\mu\nu}(x,z))\, dx^\mu dx^\nu - dz^2 \right]$$

with

$$e^{-A(z)} = \frac{1}{(1+k|z|)^2}$$

or

$$A(z) = 2 \log (k|z|+1) .$$

The relation between y and z is

$$\frac{1}{1+k|z|} = e^{-k|y|}$$

We look for linearized fluctuations around the background satisfying ("perturbed") Einstein equation,

$$\delta G_{MN} = \frac{1}{M_*^3}\, \delta T_{MN} \,.$$

We fix the so-called RS-gauge

$$h^\mu{}_\mu = \partial_\mu h^\mu{}_\nu = 0 \,,$$

and expand δG_{MN} keeping only linear terms in order to get linearized Einstein eq. in warped background,

$$-\frac{1}{2}\, \partial^R \partial_R h_{\mu\nu} + \frac{3}{4}\, \partial^R A \; \partial_R h_{\mu\nu} = 0 \,.$$

It is convenient to use the rescaling

$$h_{\mu\nu} = e^{\frac{5}{3}A}\, \tilde{h}_{\mu\nu} \,,$$

giving

$$-\frac{1}{2}\, \partial^R \partial_R \tilde{h}_{\mu\nu} + \left[\frac{9}{32}\, \partial^R A\, \partial_R A - \frac{3}{8}\, \partial^R \partial_R A \right] \tilde{h}_{\mu\nu} = 0$$

which has the form of a one-dimensional Schrödinger equation.

(Note $\partial^R \partial_R = -\Box_x - \nabla_z^2$.)

Now separate variables

$$\tilde{h}_{\mu\nu}(x,z) = \hat{h}_{\mu\nu}(x)\, \psi(z)$$

and require $\hat{h}_{\mu\nu}$ to be a 4d mass eigenstate with

$$\Box \hat{h}_{\mu\nu} = m^2 \hat{h}_{\mu\nu} .$$

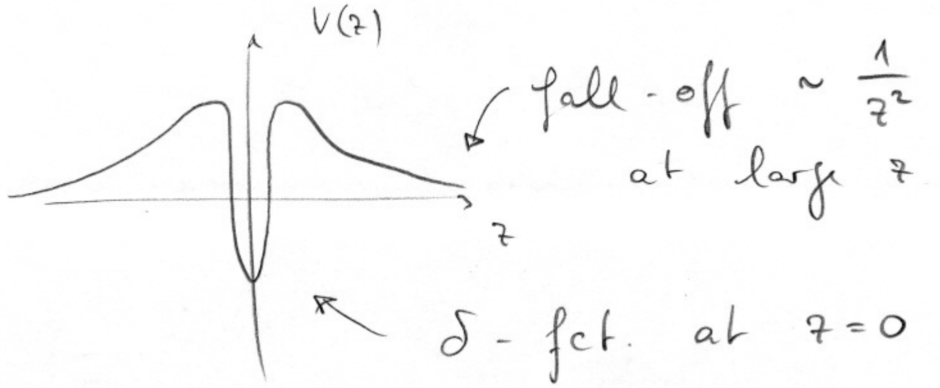
Then the Schrödinger equation for the KK modes becomes

$$-\partial_z^2 \psi + \underbrace{\left(\frac{9}{16} A'^2 - \frac{3}{4} A'' \right)}_{= V(z) \text{ Schrödinger potential}} \psi = m^2 \psi$$

We have $A = 2 \log (k|z| + 1)$, so

$$V(z) = \frac{15}{4} \frac{k^2}{(1+k|z|)^2} - \frac{3k}{1+k|z|} \delta(z) ,$$

the so-called <u>volcano potential</u>



As known from quantum mechanics the δ-fct potential has a single bound state, here: the massless graviton (which was expected since 4d Lorentz invariance is unbroken).

Explicitly, one finds solution

$$\psi^{(0)}(z) = e^{-\frac{3}{4}A(z)}$$
$$= \frac{1}{(1+k|z|)^{3/4}}$$

or in y-coordinates with

$$ds^2 = e^{-2k|y|}(\eta_{\mu\nu}+h_{\mu\nu}) - dy^2$$

(confirming the ansatz made earlier) one has

$$\psi^{(0)}(y) = e^{-\frac{3}{4}k|y|} .$$

This means that the graviton zero-mode is localized at the Planck brane ($y=0$) but exponentially suppressed ($\sim e^{-\frac{3}{4}kL}$, $kL \simeq 35$) at the TeV brane, without introducing small parameters.

→ Gravity is localized around the positive tension brane.

Here the weakness of gravity at the TeV-brane is due to the lokalization of the graviton wave function at the other brane!
(Note that this mechanism is completely different from flat (large) extra dimensions!)

Higher modes, on the other hand, are pushed to large $|z|$ by the barrier in the volcano potential and cannot easily get to the Planck brane (only by tunnelling → estimate possible with WKB methods).

To find wave functions of higher KK modes impose suitable boundary conditions obtained from orbifold conditions under $y \to -y$. For graviton that implies

$$\partial_y h_{\mu\nu} = 0 \quad \text{at } y = 0 \text{ and } L,$$

or $$\partial_z h_{\mu\nu} = 0 \quad \text{in } z\text{-coord. at branes.}$$

That gives

$$\partial_z f = -\frac{3}{2} k f \Big|_{z = z_{Pl} = 0}$$

$$\partial_z f = -\frac{3}{2} \frac{k}{1+k|z|} f \Big|_{z = z_{TeV} = \frac{1}{k} e^{kL}}$$

Then in the bulk

$$-\partial_z^2 f + \frac{15}{4} \frac{k^2}{(1+k|z|)^2} f = m^2 f ,$$

with general solution

$$f(z) = \frac{1}{\sqrt{1+k|z|}} \left[a_m Y_2\left(\frac{m(z+1)}{k}\right) + b_m J_2\left(\frac{m(z+1)}{k}\right)\right]$$

where Y_2 and J_2 are Bessel functions and a_m and b_m are determined by boundary conditions.

The mass spectrum found from this is

$$m_j = x_j \, k \, e^{-kL}$$

with x_j the roots of the Bessel function J_1, $J_1(x_j) = 0$. Approximately, they are

j	x_j
1	3.8
2	7.0
3	10.2
4	16.5
$\vdots$	$\vdots$

Note that the modes are not evenly spaced.
Since $k e^{-kL} \sim O(\text{TeV})$, the massive gravitons have masses of $O(\text{TeV})$.
Due to the barrier in the volcano potential these massive gravitons have wave functions peaked near the TeV-brane. Therefore their coupling to TeV-brane matter is exponentially enhanced.

[This can also be seen from

$$\frac{\psi(z_{\text{TeV}})}{\psi(z_{\text{Pl}})} \sim e^{kL}$$

inverse warp factor]

Therefore the interaction of TeV-brane matter with the graviton zero-mode and with the massive graviton modes is very different:

$$\mathcal{L}_{\text{TeV}}^{\text{grav-matter}} = -\frac{1}{M_{\text{Pl}}} T^{\alpha\beta} h^{(0)}_{\alpha\beta}$$

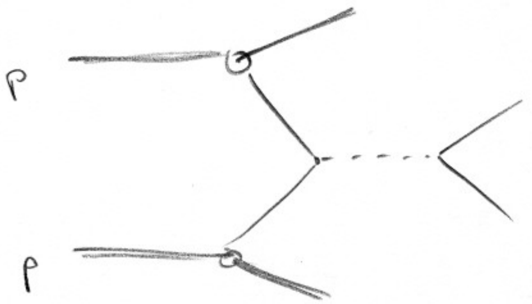
$$- \frac{1}{\underbrace{M_{\text{Pl}} e^{-kL}}_{O(\text{TeV})}} T^{\alpha\beta} \sum_{n=1}^{\infty} h^{(n)}_{\alpha\beta}$$

5) Phenomenology of RS 1

The massive gravitons couple strongly to matter on the TeV-brane. As we have seen, this holds even for individual modes. Therefore they decay quickly.

Due to the discrete spectrum with spacings $\Delta m \sim \mathcal{O}(\text{TeV})$ we can expect to see single resonances.

In hadron-hadron collisions possible channels will be for example



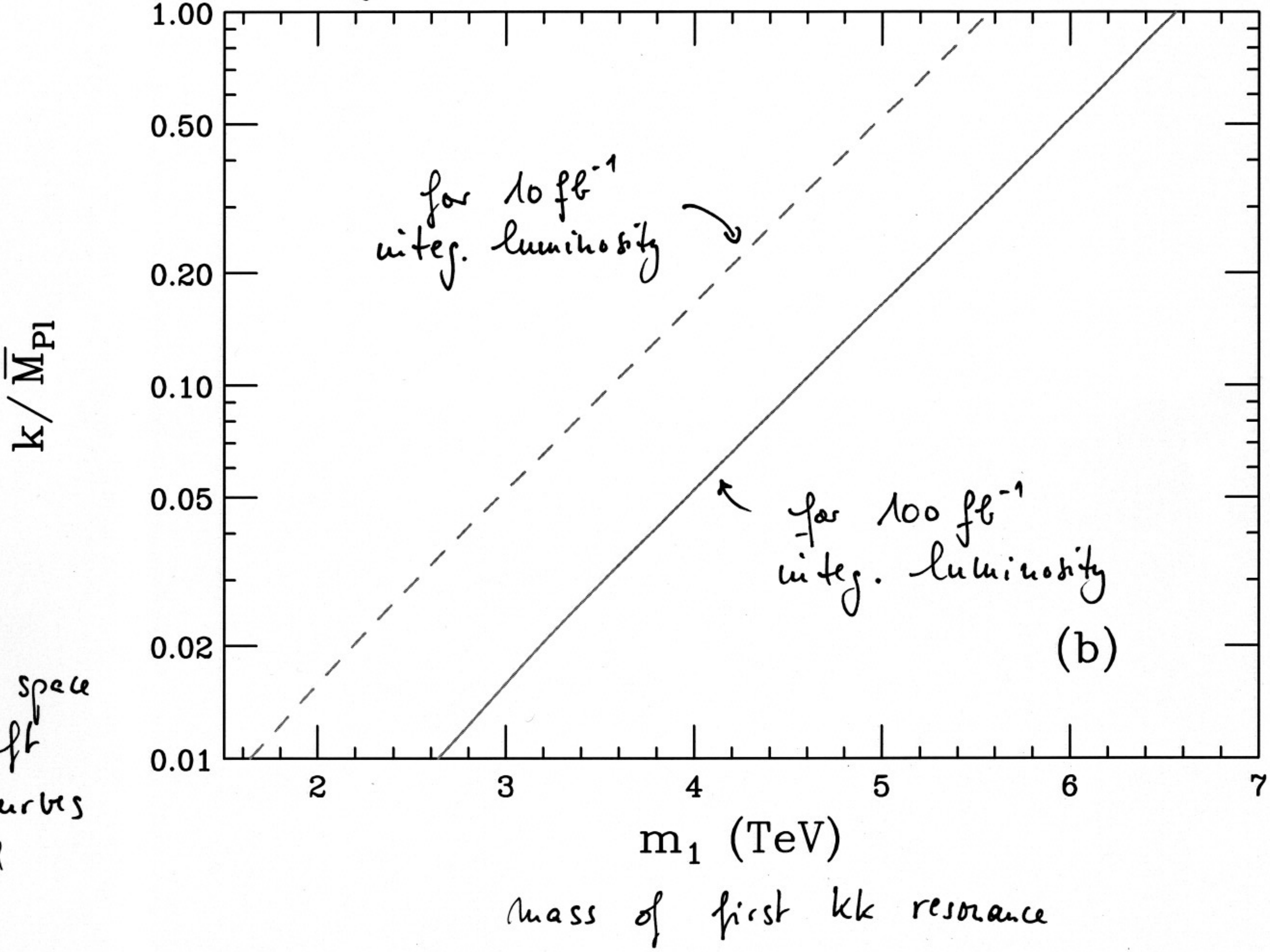
$$gg,\, q\bar{q} \to G^{(1)} \to l^+ l^-$$

$$gg,\, q\bar{q} \to G^{(1)} \to \bar{q} q,\, gg$$

corresponding to lepton pairs or 2-jet events with resonances in the invariant mass spectrum.

With these events one can test a large region in the parameter space (warped Planck scale / AdS curvature) at LHC, see figure.

LHC reach for discovery / exclusion of first massive KK graviton in RS1



parameter space to the left of the curves is covered

An even cleaner signal can be expected at a future e^+e^- - Linear Collider where broad resonances should occur in $e^+e^- \to \mu^+\mu^-$, see figure.

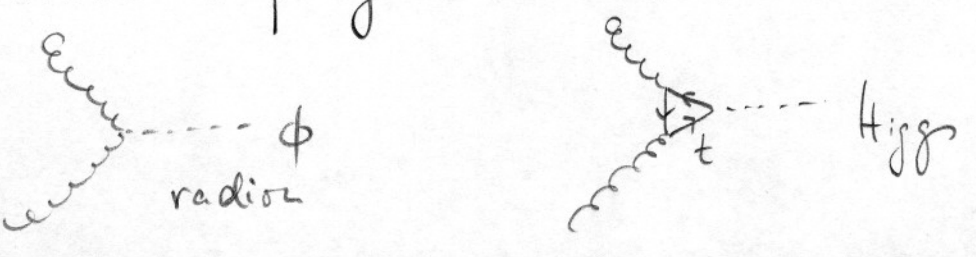
After radius stabilization the radion acquires a mass

$$m_\phi \sim \underbrace{\varepsilon}_{\mathcal{O}(1)} \frac{1}{\sqrt{kL}} M_{Pl}\, e^{-kL} .$$

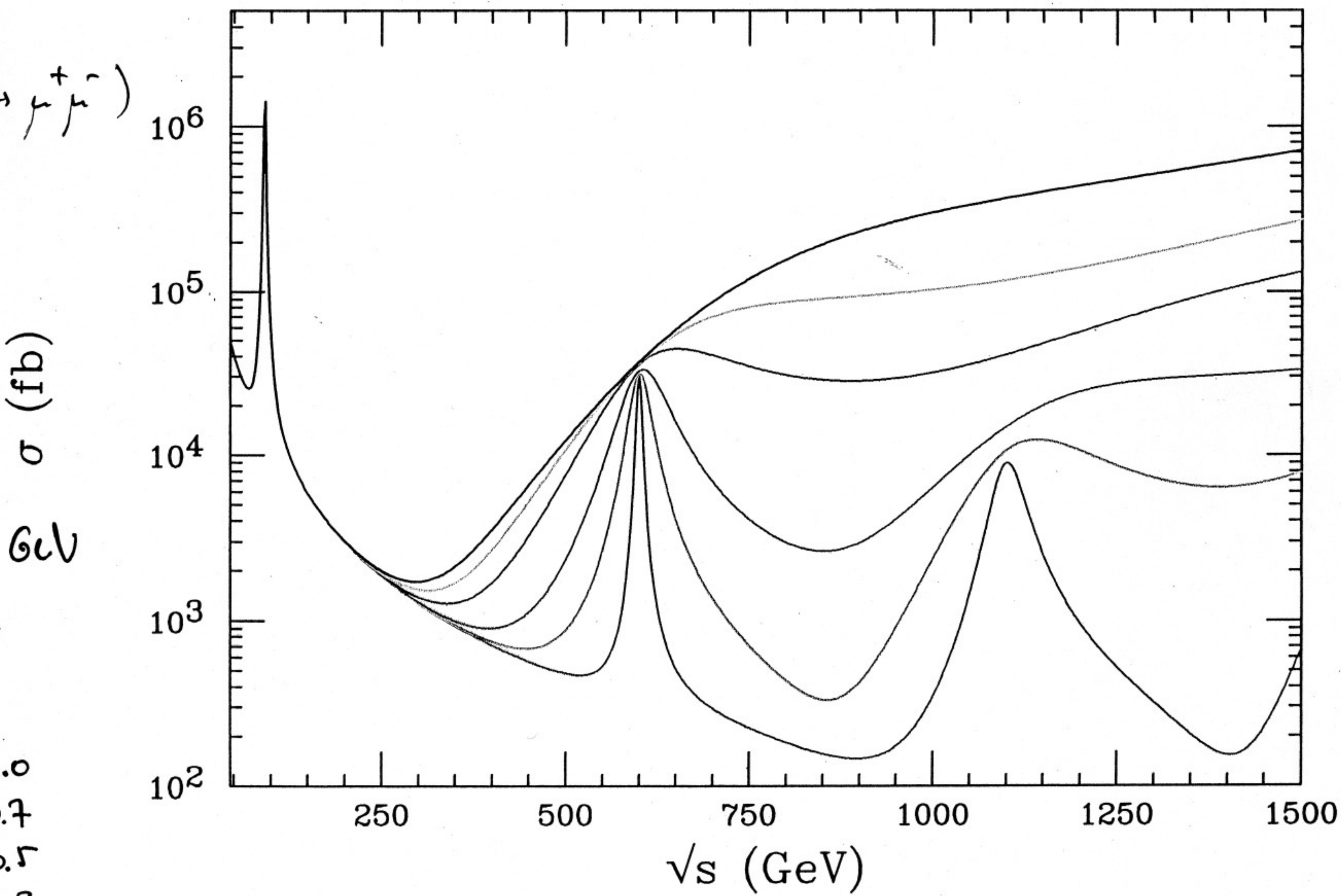
and the radion couplings are found to be

$$\frac{\phi}{\sqrt{6}\, M_{Pl}\, e^{-kL}} \, T^\mu{}_\mu$$

Many effects of the radion are <u>very similar</u> to those of a Higgs boson since they have similar couplings to matter. Radion and Higgs are therefore difficult to distinguish in RS1. Gluons, however, couple much stronger (~50 times) to radion than to Higgs, since the radion coupling is direct:



$\sigma(e^+e^- \to \mu^+\mu^-)$



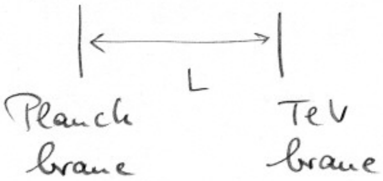
$m_1 = 600$ GeV assumed

and

$$k/M_{Pl} = \begin{cases} 1.0 \\ 0.7 \\ 0.5 \\ 0.3 \\ 0.2 \\ 0.1 \end{cases}$$

6) RS 2 model

In the RS 1 model the physical space was an interval $[0, L]$,

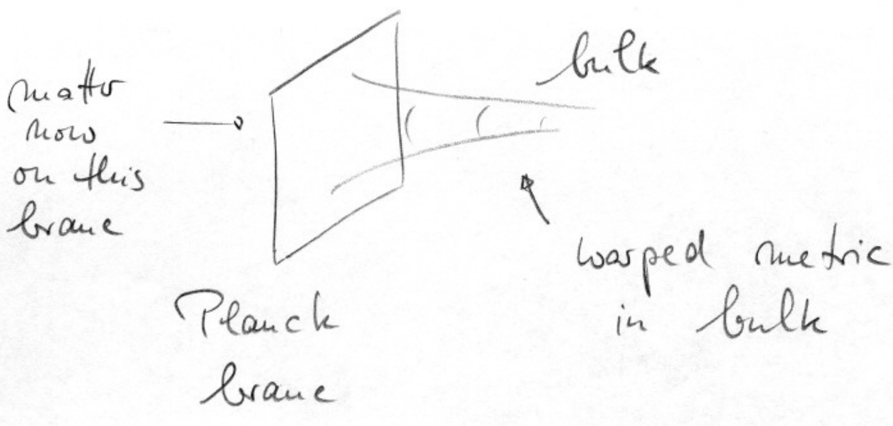


We found for the (effective) 4d Planck scale

$$M_{Pl}^2 = \frac{M_*^3}{k}\left(1 - e^{-2kL}\right).$$

Note that this depends only very little on L, and is even finite for $L \to \infty$! One could therefore consider the limit $L \to \infty$, taking the TeV brane to infinity – thus removing it completely.

Then there is only one (positive tension) brane left (the Planck brane), and we have an infinite extra dimension:



Naturally, matter now has to be put on the Planck brane.

This is the <u>RS2 model</u>.

Obviously, in contrast to the RS1 model, it is no longer designed to resolve the hierarchy problem. But it is still an interesting and simple setup.

In fact even the volume of the extra dimension remains finite.

$$V_5 = 2 \int d^4x \int_0^\infty dy \sqrt{|G|}$$
$$= V_4 \cdot 2 \cdot \int_0^\infty dy\, e^{-4ky}$$
$$= V_4 \frac{\ell}{2}$$

Hence $\ell = \frac{1}{k}$, and <u>not</u> L, plays the role of the compactification radius R_{comp} ($\rightarrow$ compare to flat extra dimensions).

Close to the brane, everything remains as it was in the RS 1 model.

In particular, the graviton zero-mode remains localized to the Planck brane, since it remains normalizable:
We had

$$\psi^{(0)} = e^{-\frac{3}{4}A(z)}$$

$$\rightarrow \int_0^{z_0} dz\, |\psi^{(0)}|^2 = \int_0^{z_0} dz\, e^{-\frac{3}{2}A(z)}$$

$$= \int_0^{z_0} dz\, \frac{1}{(1+k|z|)^{3/2}}$$

which is convergent in the limit $z_0 \to \infty$ ($\hat{=}\ L \to \infty$).

[Recall that in flat extra dimensions the zero-mode would become non-normalizable for $R_{comp} \to \infty$ due to the infinite volume of the flat extra dim and would then decouple.]

Again, the massive gravitons will stay away from the brane.
Therefore, normal 4d gravity is recovered on the Planck brane.

More precisely, corrections to the Newton potential in the effective 4d theory on the brane are of the form

$$V(r) = G_N \frac{M_1 M_2}{r} \left(1 + \frac{C}{(kr)^2}\right)$$

with $C \sim O(1)$. Since $k \sim M_{Pl}$ these corrections will be tiny.

One can even show that not only the Newton potential but also full 4d Einstein gravity is recovered on the brane in RS2.

Experimentally, it will be very difficult to distinguish RS2 from the SM. On the brane everything looks like the SM plus gravity. Therefore there is probably no chance to find limits for RS2 at colliders.

But there might be realistic possibilities to find limits for RS2 in astrophysics and (more likely) in cosmology.

7) AdS / CFT

It has been conjectured by Maldacena (and subsequently been confirmed in many tests) that gravity in AdS_5 space is equivalent to a certain gauge theory on 4d Minkowski space.
More precisely, type IIB superstring theory on AdS_5 is equivalent to maximally supersymmetric Yang-Mills theory in 4d, according to that conjecture.
This supersymmetric theory in 4d is scale invariant and hence a conformal field theory (CFT).
The Maldacena conjecture has been extended in various ways and indeed seems to indicate much deeper relations between theories in different dimensions, in particular it relates theories with strong coupling to theories with weak coupling.

As the Randall - Sundrum models live exactly in AdS_5 the conjecture can be used to study these models. What emerges is an interesting <u>holographic picture</u> of warped extra dimensions, in which the physics in the bulk is "holographically" represented by the physics on the brane.

This holographic point of view has by now become very common in studies of RS models.

Details of the AdS / CFT correspondence are beyond the scope of the present lectures, but are highly recommended for further private study.