COVARIANT SUPERGRAPHS

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Content

$\mathcal{N}=1$ super Yang-Mills theory

$$S = \int \mathrm{d}^8 z\, \phi^\dagger\, \mathrm{e}^V\, \phi + \frac{1}{g^2} \int \mathrm{d}^6 z\, \mathrm{tr}_\mathrm{F}\, (\mathcal{W}^\alpha \mathcal{W}_\alpha) + \left\{ \int \mathrm{d}^6 z\, \mathcal{P}(\phi) \ + \ \text{c.c.} \right\} .$$

Here $\mathcal{P}(\phi)$ is the superpotential,

$$\mathcal{P}(\phi) = \frac{1}{2} \mu_{ij}\, \phi^i \phi^j + \frac{1}{6} \lambda_{ijk}\, \phi^i \phi^j \phi^k \ ,$$

with μ_{ij} and λ_{ijk} invariant tensors of the gauge group.

`Supersymmetric matter:` Chiral superfield (scalar multiplet) ϕ, $\bar{D}_{\dot\alpha} \phi = 0$, transforms in a representation R of the gauge group.

$$\phi = \Big(\phi^i(z) \Big) \ , \qquad \phi^\dagger = \Big(\bar{\phi}_i(z) \Big) \ .$$

`Supersymmetric gauge field` (vector multiplet):

$$V = V^I(z)\, T_I = V^\dagger \ ,$$
$$\mathcal{W}_\alpha = -\frac{1}{8} \bar{D}^2 (\mathrm{e}^{-V}\, D_\alpha\, \mathrm{e}^V \cdot 1) \ ,$$

with T_I the generators of the gauge group. The generators are normalized such that $\mathrm{tr}_\mathrm{F}\, (T_I\, T_J) = \delta_{IJ}$ in the fundamental (defining) representation of the gauge group. In what follows, $\mathrm{tr}_\mathrm{F} = \mathrm{tr}$.

$D_A = (\partial_a, D_\alpha, \bar{D}^{\dot\alpha})$ are the flat superspace covariant derivatives,

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + \mathrm{i}\, (\sigma^b)_{\alpha\dot\beta}\, \bar\theta^{\dot\beta}\, \partial_b \ , \qquad \bar{D}^{\dot\alpha} = \frac{\partial}{\partial \bar\theta_{\dot\alpha}} + \mathrm{i}\, (\tilde\sigma^b)^{\dot\alpha\beta}\, \theta_\beta\, \partial_b \ ,$$
$$[D_A, D_B\} = T_{AB}{}^C\, D_C \ .$$

Interesting sectors of low-energy effective actions

(i) Kähler potential

$$\int \mathrm{d}^8 z\, K(\phi, \phi^\dagger\, \mathrm{e}^V)\ ;$$

(ii) Effective gauge kinetic term

$$\int \mathrm{d}^6 z\, f_{IJ}(\phi)\, \mathcal{W}^{I\alpha}\, \mathcal{W}^J_\alpha\ ;$$

(iii) Euler-Heisenberg-type actions (for a $U(1)$ vector multiplet)

$$\int \mathrm{d}^6 z\, \mathcal{W}^2 + \int \mathrm{d}^8 z\, \mathcal{W}^2 \bar{\mathcal{W}}^2\, \Lambda(D^2\mathcal{W}^2, \bar{D}^2\bar{\mathcal{W}}^2);$$

More general (superconformally invariant) action

$$\int \mathrm{d}^6 z\, \mathcal{W}^2 + \int \mathrm{d}^8 z\, \frac{\mathcal{W}^2 \bar{\mathcal{W}}^2}{(\bar{\phi}\phi)^2}\, \Lambda\Big(\frac{D^2\mathcal{W}^2}{(\bar{\phi}\phi)^2}\, ,\, \frac{\bar{D}^2\bar{\mathcal{W}}^2)}{(\bar{\phi}\phi)^2}\Big)\ .$$

One needs powerful diagram techniques in order to compute loop quantum corrections to such low-energy actions.

`Notation:`

Superspace coordinates: $z^A = (x^a, \theta^\alpha, \bar{\theta}_{\dot{\alpha}})$

Superspace integration measures:

$$\mathrm{d}^8 z = \mathrm{d}^4 x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\ , \qquad \mathrm{d}^6 z = \mathrm{d}^4 x\, \mathrm{d}^2\theta\ , \qquad \mathrm{d}^6 \bar{z} = \mathrm{d}^4 x\, \mathrm{d}^2\bar{\theta}\ .$$

Two frames in $\mathcal{N}=1$ SYM: τ-frame and λ-frame

τ-frame

The vector multiplet is described by gauge-covariant derivatives

$$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^{\dot\alpha}) = D_A + \mathrm{i}\,\Gamma_A\ , \qquad \Gamma_A = \Gamma^I_A(z)\,T_I\ ,$$
$$[\mathcal{D}_A, \mathcal{D}_B\} = T_{AB}{}^C\,\mathcal{D}_C + \mathrm{i}\,\mathcal{F}_{AB}(z)\ , \qquad \mathcal{F}_{AB} = \mathcal{F}^I_{AB}(z)\,T_I\ ,$$

with Γ_A the connection taking its values in the Lie algebra of the gauge group.

Gauge transformation laws:

$$\mathcal{D}_A \ \to\ \mathrm{e}^{\mathrm{i}\tau(z)}\,\mathcal{D}_A\,\mathrm{e}^{-\mathrm{i}\tau(z)}\ , \qquad \Psi\ \to\ \mathrm{e}^{\mathrm{i}\tau(z)}\,\Psi\ , \qquad \tau^\dagger = \tau\ ,$$

with Ψ a matter multiplet. The gauge parameter $\tau = \tau^I(z)\,T_I$ is arbitrary modulo the reality condition imposed.

The gauge covariant derivatives constitute the following algebra:

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} = 0\ , \qquad \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\beta}\} = -2\mathrm{i}\,\mathcal{D}_{\alpha\dot\beta}\ ,$$
$$[\mathcal{D}_\alpha, \mathcal{D}_{\beta\dot\beta}] = 2\mathrm{i}\varepsilon_{\alpha\beta}\,\bar{\mathcal{W}}_{\dot\beta}\ , \qquad [\bar{\mathcal{D}}_{\dot\alpha}, \mathcal{D}_{\beta\dot\beta}] = 2\mathrm{i}\varepsilon_{\dot\alpha\dot\beta}\,\mathcal{W}_\beta\ ,$$
$$[\mathcal{D}_{\alpha\dot\alpha}, \mathcal{D}_{\beta\dot\beta}] = \mathrm{i}\,\mathcal{F}_{\alpha\dot\alpha,\beta\dot\beta} = -\varepsilon_{\alpha\beta}\,\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{W}}_{\dot\beta} - \varepsilon_{\dot\alpha\dot\beta}\,\mathcal{D}_\alpha\mathcal{W}_\beta\ .$$

The spinor field strengths $\mathcal{W}_\alpha$ and $\bar{\mathcal{W}}_{\dot\alpha} = (\mathcal{W}_\alpha)^\dagger$ obey the Bianchi identities

$$\bar{\mathcal{D}}_{\dot\alpha}\mathcal{W}_\alpha = \mathcal{D}_\alpha\bar{\mathcal{W}}_{\dot\alpha} = 0\ , \qquad \mathcal{D}^\alpha\mathcal{W}_\alpha = \bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha}\ .$$

`Solution to the constraints:`

$$\mathcal{D}_\alpha = \mathrm{e}^{-\Omega}\, D_\alpha\, \mathrm{e}^{\Omega}~, \qquad \bar{\mathcal{D}}_{\dot\alpha} = \mathrm{e}^{\Omega^\dagger}\, \bar{D}_{\dot\alpha}\, \mathrm{e}^{-\Omega^\dagger}~,$$

with $\Omega = \Omega^I(z)\, T_I$ and $\Omega^\dagger = \bar{\Omega}^I(z)\, T_I$ the so-called *prepotentials*.

Historical comment: The term "pre-potential" was introduced by S. J. Gates and W. Siegel in 1980.

The `gauge transformation laws` of $\Omega^\dagger$ and Ω:

$$\begin{aligned} \mathrm{e}^{\Omega^\dagger} &\to \mathrm{e}^{\mathrm{i}\tau}\, \mathrm{e}^{\Omega^\dagger}\, \mathrm{e}^{-\mathrm{i}\lambda}~, \qquad \bar{D}_{\dot\alpha}\lambda = 0~, \qquad \lambda = \lambda^I(z)\, T_I~, \\ \mathrm{e}^{-\Omega} &\to \mathrm{e}^{\mathrm{i}\tau}\, \mathrm{e}^{-\Omega}\, \mathrm{e}^{-\mathrm{i}\lambda^\dagger}~, \qquad D_\alpha \lambda^\dagger = 0~. \end{aligned}$$

Covariantly chiral superfield

$$\bar{\mathcal{D}}_{\dot\alpha}\Phi = 0 \quad \Longleftrightarrow \quad \Phi = \mathrm{e}^{\Omega^\dagger}\, \phi~, \qquad \bar{D}_{\dot\alpha}\phi = 0~.$$

The `gauge transformation laws` of Φ and ϕ:

$$\Phi \to \mathrm{e}^{\mathrm{i}\tau}\, \Phi~, \qquad \phi \to \mathrm{e}^{\mathrm{i}\lambda}\, \phi~.$$

$\boldsymbol{\lambda}$-frame

(covariantly chiral representation)

$$\mathcal{D}_A \;\to\; \mathrm{e}^{-\Omega^\dagger}\,\mathcal{D}_A\,\mathrm{e}^{\Omega^\dagger}\;, \qquad \Psi \;\to\; \mathrm{e}^{-\Omega^\dagger}\,\Psi\;.$$

The `gauge transformation law` of $\mathcal{D}_A$ in the λ-frame:

$$\mathcal{D}_A \;\to\; \mathrm{e}^{\mathrm{i}\lambda}\,\mathcal{D}_A\,\mathrm{e}^{-\mathrm{i}\lambda}\;, \qquad \Psi \;\to\; \mathrm{e}^{\mathrm{i}\lambda}\,\Psi\;, \qquad \bar{D}_{\dot\alpha}\lambda = 0\;.$$

In this frame, the covariant derivatives are:

$$\mathcal{D}_\alpha = \mathrm{e}^{-V}\,D_\alpha\,\mathrm{e}^{V}\;, \qquad \bar{\mathcal{D}}_{\dot\alpha} = \bar{D}_{\dot\alpha}\;,$$

with

$$\mathrm{e}^{V} = \mathrm{e}^{\Omega}\,\mathrm{e}^{\Omega^\dagger}\;, \qquad V^\dagger = V\;.$$

Covariantly chiral and antichiral superfields in the λ-frame:

$$\Phi = \phi\;, \qquad \Phi^\dagger = \phi^\dagger\,\mathrm{e}^{V}\;.$$

The `gauge transformation law` of V:

$$\mathrm{e}^{V} \;\to\; \mathrm{e}^{\mathrm{i}\lambda^\dagger}\,\mathrm{e}^{V}\,\mathrm{e}^{-\mathrm{i}\lambda}\;.$$

In the λ-frame, the τ-gauge freedom is absent (under the τ-transformations, $\delta\Omega = -\mathrm{i}\,\tau + O(\Omega)$, and therefore $\mathrm{Im}\,\Omega$ can be completely gauged away).

In what follows, we do not distinguish between Φ and ϕ.

$\mathcal{N} = 1$ SYM

$$S = \int \mathrm{d}^8 z\, \Phi^\dagger \Phi + \frac{1}{g^2} \int \mathrm{d}^6 z\, \mathrm{tr}\,(\mathcal{W}^\alpha \mathcal{W}_\alpha) + \left\{ \int \mathrm{d}^6 z\, \mathcal{P}(\Phi) \; + \; \mathrm{c.c.} \right\} \;,$$

$$\mathcal{P}(\Phi) = \frac{1}{2}\mu_{ij}\, \Phi^i \Phi^j + \frac{1}{6}\lambda_{ijk}\, \Phi^i \Phi^j \Phi^k \;.$$

Our consideration will be restricted to the special case:

$\mathcal{N} = 2$ SYM ($R \; \rightarrow \; \mathrm{Ad} \oplus \mathrm{R} \oplus \bar{\mathrm{R}}$)

$$\Phi^i \quad \Longrightarrow \quad \begin{pmatrix} \Phi^I \\ Q^i \\ \tilde{Q}_i \end{pmatrix}$$

Action functional:

$$S = S_{\mathrm{SYM}} + S_{\mathrm{hyper}} \;,$$

$$S_{\mathrm{SYM}} = \frac{1}{g^2}\, \mathrm{tr} \Big(\int \mathrm{d}^8 z\, \Phi^\dagger \Phi + \int \mathrm{d}^6 z\, \mathcal{W}^\alpha \mathcal{W}_\alpha \Big) \equiv S_{\mathrm{scal}} + S_{\mathrm{vect}} \;,$$

$$S_{\mathrm{hyper}} = \int \mathrm{d}^8 z\, (\mathcal{Q}^\dagger\, \mathcal{Q} + \tilde{\mathcal{Q}} \tilde{\mathcal{Q}}^\dagger) - \mathrm{i} \int \mathrm{d}^6 z\, \tilde{\mathcal{Q}} \Phi \mathcal{Q} + \mathrm{i} \int \mathrm{d}^6 \bar{z}\, \mathcal{Q}^\dagger \Phi^\dagger \tilde{\mathcal{Q}}^\dagger \;.$$

`Massive case` is equivalent to coupling to a “frozen” Abelian $\mathcal{N} = 2$ vector multiplet:

Gauge group $G \; \rightarrow \; G \times U(1)$

Adjoint chiral multiplet $\Phi \; \rightarrow \; \Phi + \mu\, \mathbf{1}, \qquad \mu = \mathrm{const}$

$\mathcal{N} = 1$ vector multiplet $\mathcal{W}_\alpha \; \rightarrow \; \mathcal{W}_\alpha$.

`Quantum superconformal (finite) theories`:

$$\mathrm{tr}_{\mathrm{Ad}}\, \Phi^2 \; = \; \mathrm{tr}_{\mathrm{R}}\, \Phi^2 \;.$$

Background field quantization

Grisaru, Roček, Siegel (1979)

Split the dynamical variables into background and quantum ones,

$$\Phi \;\to\; \Phi + \varphi \;, \qquad \mathcal{Q} \;\to\; \mathcal{Q} + q \;, \qquad \tilde{\mathcal{Q}} \;\to\; \tilde{\mathcal{Q}} + \tilde{q} \;,$$
$$\mathcal{D}_\alpha \;\to\; \mathrm{e}^{-v}\, \mathcal{D}_\alpha\, \mathrm{e}^{v} \;, \qquad \bar{\mathcal{D}}_{\dot\alpha} \;\to\; \bar{\mathcal{D}}_{\dot\alpha} \;,$$

with lower-case letters used for the quantum superfields. We will *not* be interested in the dependence of the effective action on the hypermultiplet superfields, and $\mathcal{Q} = \tilde{\mathcal{Q}} = 0$ in what follows.

The action $S_{\rm SYM}$ turns into

$$S_{\rm SYM} = \frac{1}{g^2}\,{\rm tr}\left(\int {\rm d}^8 z\, (\Phi + \varphi)^\dagger\, \mathrm{e}^{v}\, (\Phi + \varphi)\, \mathrm{e}^{-v} + \int {\rm d}^6 z\, \mathbf{W}^\alpha \mathbf{W}_\alpha \right) \;,$$

where

$$\mathbf{W}_\alpha = -\frac{1}{8}\bar{\mathcal{D}}^2(\mathrm{e}^{-v}\, \mathcal{D}_\alpha\, \mathrm{e}^{v} \cdot 1) = \mathcal{W}_\alpha - \frac{1}{8}\bar{\mathcal{D}}^2\Big(\mathcal{D}_\alpha v - \frac{1}{2}[v, \mathcal{D}_\alpha v]$$
$$+\frac{1}{6}[v,[v,\mathcal{D}_\alpha v]] - \frac{1}{24}[v,[v,[v,\mathcal{D}_\alpha v]]]\Big) + O(v^5) \;.$$

The hypermultiplet action takes the form

$$S_{\rm hyper} = \int {\rm d}^8 z\, (q^\dagger\, \mathrm{e}^{v}\, q + \tilde{q}\, \mathrm{e}^{-v}\, \tilde{q}^\dagger)$$
$$- {\rm i} \int {\rm d}^6 z\, \tilde{q}(\Phi + \varphi) q + {\rm i} \int {\rm d}^6 \bar{z}\, q^\dagger (\Phi + \varphi)^\dagger \tilde{q}^\dagger \;.$$

Appendix: Technical details

Background-quantum splitting:

$$\mathrm{e}^{-\Omega} = \mathrm{e}^{-\Omega_Q}\,\mathrm{e}^{-\Omega_B}\ , \qquad \mathrm{e}^{\Omega} = \mathrm{e}^{\Omega_B}\,\mathrm{e}^{\Omega_Q}\ ,$$
$$\mathrm{e}^{\Omega^\dagger} = \mathrm{e}^{\Omega_Q^\dagger}\,\mathrm{e}^{\Omega_B^\dagger}\ , \qquad \mathrm{e}^{-\Omega^\dagger} = \mathrm{e}^{-\Omega_B^\dagger}\,\mathrm{e}^{-\Omega_Q^\dagger}\ .$$

Covariant derivatives:

$$\mathcal{D}_\alpha = \mathrm{e}^{-\Omega_Q}\,\mathrm{e}^{-\Omega_B}\,D_\alpha\,\mathrm{e}^{\Omega_B}\,\mathrm{e}^{\Omega_Q} \equiv \mathrm{e}^{-\Omega_Q}\,\nabla_\alpha \mathrm{e}^{\Omega_Q}\ ,$$
$$\bar{\mathcal{D}}_{\dot\alpha} = \mathrm{e}^{\Omega_Q^\dagger}\,\mathrm{e}^{\Omega_B^\dagger}\,\bar D_{\dot\alpha}\,\mathrm{e}^{-\Omega_B^\dagger}\,\mathrm{e}^{-\Omega_Q^\dagger} \equiv \mathrm{e}^{\Omega_Q^\dagger}\,\bar\nabla\,\mathrm{e}^{-\Omega_Q^\dagger}\ .$$

$\nabla_A = (\nabla_a, \nabla_\alpha, \bar\nabla^{\dot\alpha})$ the background gauge-covariant derivatives.

For a `covariantly chiral superfield`

$$\Psi = \mathrm{e}^{\Omega^\dagger}\,\psi\ , \qquad \bar{\mathcal{D}}_{\dot\alpha}\Psi = 0 \quad \Longleftrightarrow \quad \bar D_{\dot\alpha}\psi = 0\ ,$$

we get

$$\Psi = \mathrm{e}^{\Omega_Q^\dagger}\,\mathrm{e}^{\Omega_B^\dagger}\,\psi \equiv \mathrm{e}^{\Omega_Q^\dagger}\,\boldsymbol{\psi}\ , \qquad \bar\nabla_{\dot\alpha}\boldsymbol{\psi} = 0\ .$$

$\boldsymbol{\psi}$ background covariantly chiral superfield.

Background gauge freedom:

$$\begin{array}{rcl} e^{\Omega_B^\dagger} & \longrightarrow & e^{i\tau_B}\, e^{\Omega_B^\dagger}\, e^{-i\lambda_B} \ , \qquad \bar{D}_{\dot\alpha}\lambda_B = 0 \ , \\ e^{\Omega_Q^\dagger} & \longrightarrow & e^{i\tau_B}\, e^{\Omega_Q^\dagger}\, e^{-i\tau_B} \ , \\ \boldsymbol{\psi} & \longrightarrow & e^{i\tau_B}\, \boldsymbol{\psi} \ . \end{array}$$

Quantum gauge freedom:

$$\begin{array}{rcl} e^{\Omega_B^\dagger} & \longrightarrow & e^{\Omega_B^\dagger} \ , \\ e^{\Omega_Q^\dagger} & \longrightarrow & e^{i\tau_Q}\, e^{\Omega_Q^\dagger}\, e^{-i\lambda_Q} \ , \qquad \bar{\nabla}_{\dot\alpha}\lambda_Q = 0 \ , \\ \boldsymbol{\psi} & \longrightarrow & e^{i\lambda_Q}\, \boldsymbol{\psi} \ . \end{array}$$

Introduce `quantum gauge field`

$$e^v = e^{\Omega_Q}\, e^{\Omega_Q^\dagger} \ .$$

Background gauge freedom:

$$e^v \longrightarrow e^{i\tau_B}\, e^v\, e^{-i\tau_B} \ .$$

Quantum gauge freedom:

$$e^v \longrightarrow e^{i\lambda_Q^\dagger}\, e^v\, e^{-i\lambda_Q} \ .$$

Quantum $\boldsymbol{\lambda}$-frame

(quantum chiral representation)

$$\begin{array}{rcl} \mathcal{D}_A & \rightarrow & e^{-\Omega_Q^\dagger}\, \mathcal{D}_A\, e^{\Omega_Q^\dagger} \quad \Longleftrightarrow \quad \mathcal{D}_\alpha = e^{-v}\, \nabla_\alpha\, e^v \ , \quad \bar{\mathcal{D}}_{\dot\alpha} = \bar{\nabla}_{\dot\alpha} \ , \\ \Psi & \rightarrow & e^{-\Omega_Q^\dagger}\, \Psi \ . \end{array}$$

In what follows, we do not distinguish between $\mathcal{D}_A$ and ∇_A.

Supersymmetric 't Hooft gauge (a special case of the supersymmetric R_ξ-gauge)

Ovrut, Wess (82)

Banin, Buchbinder, Pletnev (2002)

`Nonlocal` gauge conditions (to eliminate the v-φ-mixing):

$$-4\chi = \bar{\mathcal{D}}^2 v + [\Phi, (\Box_+)^{-1}\bar{\mathcal{D}}^2\varphi^\dagger] = \bar{\mathcal{D}}^2 v + [\Phi, \bar{\mathcal{D}}^2(\Box_-)^{-1}\varphi^\dagger] ,$$
$$-4\chi^\dagger = \mathcal{D}^2 v - [\Phi^\dagger, (\Box_-)^{-1}\mathcal{D}^2\varphi] = \mathcal{D}^2 v - [\Phi^\dagger, \mathcal{D}^2(\Box_+)^{-1}\varphi] .$$

Here $\Box_+$ is the *covariantly chiral d'Alembertian*,

$$\Box_+ = \mathcal{D}^a\mathcal{D}_a - \mathcal{W}^\alpha\mathcal{D}_\alpha - \frac{1}{2}\,(\mathcal{D}^\alpha\mathcal{W}_\alpha) ,$$
$$\Box_+\Psi = \frac{1}{16}\,\bar{\mathcal{D}}^2\mathcal{D}^2\Psi , \quad \bar{\mathcal{D}}_{\dot\alpha}\Psi = 0 ,$$

for a covariantly chiral superfield Ψ.

Similarly, $\Box_-$ is the *covariantly antichiral d'Alembertian*,

$$\Box_- = \mathcal{D}^a\mathcal{D}_a + \bar{\mathcal{W}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} + \frac{1}{2}\,(\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha}) ,$$
$$\Box_-\bar\Psi = \frac{1}{16}\,\mathcal{D}^2\bar{\mathcal{D}}^2\bar\Psi , \quad \mathcal{D}_\alpha\bar\Psi = 0 ,$$

for a covariantly antichiral superfield $\bar\Psi$.

Important properties:

$$\mathcal{D}^2\,\Box_+ = \Box_-\,\mathcal{D}^2 , \qquad \bar{\mathcal{D}}^2\,\Box_- = \Box_+\,\bar{\mathcal{D}}^2 .$$

The gauge conditions lead to the *Faddeev-Popov ghost action*

$$S_{\rm gh} = {\rm tr} \int {\rm d}^8 z\, (\tilde{c} - \tilde{c}^\dagger) \left\{ L_{v/2}\, (c + c^\dagger) + L_{v/2}\, \coth(L_{v/2})(c - c^\dagger) \right\}$$
$$- {\rm tr} \int {\rm d}^8 z\, \left\{ [\tilde{c}, \Phi]\, (\Box_-)^{-1} [c^\dagger, \Phi^\dagger + \varphi^\dagger] + [\tilde{c}^\dagger, \Phi^\dagger]\, (\Box_+)^{-1} [c, \Phi + \varphi] \right\} ,$$

where $L_X\, Y = [X, Y]$. Here the anticommuting ghost superfields, c and $\tilde{c}$, are *background covariantly chiral.*

Useful *gauge-fixing functional*:

$$S_{\rm gf} = -{\rm tr} \int {\rm d}^8 z\, \chi^\dagger\, \chi\ .$$

The quantum quadratic part of $S_{\rm SYM} + S_{\rm gf}$ is

$$S^{(2)}_{\rm SYM} + S_{\rm gf} = {\rm tr} \int {\rm d}^8 z\, (\varphi^\dagger\, \varphi - [\Phi^\dagger, [\Phi, \varphi^\dagger]]\, (\Box_+)^{-1}\, \varphi)$$
$$- \frac{1}{2}\, {\rm tr} \int {\rm d}^8 z\, v\, (\Box_{\rm v} v - [\Phi^\dagger, [\Phi, v]]) \ + \ \dots$$

where the dots stand for the terms with derivatives of the background (anti)chiral superfields $\Phi^\dagger$ and Φ.

$\Box_{\rm v}$ is the *vector d'Alembertian,*

$$\begin{aligned}
\Box_{\rm v} &= \mathcal{D}^a \mathcal{D}_a - \mathcal{W}^\alpha \mathcal{D}_\alpha + \bar{\mathcal{W}}_{\dot\alpha} \bar{\mathcal{D}}^{\dot\alpha} \\
&= -\frac{1}{8} \mathcal{D}^\alpha \bar{\mathcal{D}}^2 \mathcal{D}_\alpha + \frac{1}{16} \{\mathcal{D}^2, \bar{\mathcal{D}}^2\} - \mathcal{W}^\alpha \mathcal{D}_\alpha - \frac{1}{2} (\mathcal{D}^\alpha \mathcal{W}_\alpha) \\
&= -\frac{1}{8} \bar{\mathcal{D}}_{\dot\alpha} \mathcal{D}^2 \bar{\mathcal{D}}^{\dot\alpha} + \frac{1}{16} \{\mathcal{D}^2, \bar{\mathcal{D}}^2\} + \bar{\mathcal{W}}_{\dot\alpha} \bar{\mathcal{D}}^{\dot\alpha} + \frac{1}{2} (\bar{\mathcal{D}}_{\dot\alpha} \bar{\mathcal{W}}^{\dot\alpha})\ .
\end{aligned}$$

The gauge-fixing functional chosen is accompanied by the presence of the *Nielsen-Kallosh ghost action*

$$S_{\rm NK} = {\rm tr} \int {\rm d}^8 z \, b^\dagger \, b \ ,$$

where the anticommuting *third ghost* superfield b is background covariantly chiral. The Nielsen-Kallosh ghosts lead to a *one-loop contribution* only.

The background superfields will be chosen to form a special on-shell $\mathcal{N} = 2$ vector multiplet in the Cartan subalgebra of the gauge group:

$$[\Phi, \bar{\Phi}] \ = \ \mathcal{D}^\alpha \mathcal{W}_\alpha = 0 \ , \qquad \mathcal{D}_\alpha \Phi \ = \ 0 \ .$$

Such a background configuration is convenient for computing those corrections to the effective action which do not contain derivatives of Φ and $\Phi^\dagger$.

The quantum quadratic part of the action $S_{\rm hyper}$ is

$$S^{(2)}_{\rm hyper} = \int {\rm d}^8 z\,(q^\dagger\, q + \tilde q\,\tilde q^\dagger) + \int {\rm d}^6 z\,\tilde q\,\mathcal{M}_{\rm R}\, q + \int {\rm d}^6 \bar z\, q^\dagger\,\mathcal{M}^\dagger_{\rm R}\,\tilde q^\dagger\ .$$

Here the "mass" operator $\mathcal{M}$ is defined by

$$\mathcal{M}_{\rm R}\,\Sigma = -{\rm i}\,\Phi\,\Sigma\ ,$$

for a multiplet Σ transforming in the representation R.

The action $S^{(2)}_{\rm SYM} + S_{\rm GF}$ becomes

$$S^{(2)}_{\rm SYM} + S_{\rm GF} = \frac{1}{g^2}\,{\rm tr}\int {\rm d}^8 z\,\Big(\varphi^\dagger\frac{1}{\Box_+}(\Box_+ - |\mathcal{M}_{\rm Ad}|^2)\varphi \\ -\frac{1}{2}v(\Box_{\rm v} - |\mathcal{M}_{\rm Ad}|^2)v\Big)\ .$$

The qudratic part of the Faddeev-Popov ghost action becomes

$$S^{(2)}_{\rm gh} = {\rm tr}\int {\rm d}^8 z\,\Big(c^\dagger(\Box_+)^{-1}(\Box_+ - |\mathcal{M}_{\rm Ad}|^2)\tilde c \\ -\tilde c^\dagger(\Box_+)^{-1}(\Box_+ - |\mathcal{M}_{\rm Ad}|^2)c\Big)\ .$$

The adjoint "mass" matrix:

$$\mathcal{M}_{\rm Ad}\,\Sigma = -{\rm i}\,[\Phi,\Sigma]\ ,\qquad |\mathcal{M}_{\rm Ad}|^2\,\Sigma = [\Phi^\dagger,[\Phi,\Sigma]] = [\Phi,[\Phi^\dagger,\Sigma]]\ .$$

Covariant Feynman propagators

All the Feynman propagators associated with the actions

$$S^{(2)}_{\rm SYM} + S_{\rm GF}~, \qquad S^{(2)}_{\rm gh}~, \qquad S^{(2)}_{\rm hyper}~,$$

can be expressed via a single Green's function in different representations of the gauge group. Such a Green's function, $G^{(\rm R)}(z,z')$, originates in the following auxiliary model

$$S^{(\rm R)} = \int {\rm d}^8 z\, \Sigma^\dagger (\Box_{\rm v} - |\mathcal{M}_{\rm D}|^2) \Sigma~,$$

which describes the dynamics of an unconstrained complex superfield Σ transforming in the representation R of the gauge group. The relevant Feynman propagator reads

$$G^{(\rm R)}(z,z') = {\rm i}\, \langle 0| T\, (\Sigma(z)\, \Sigma^\dagger(z'))|0\rangle \equiv {\rm i}\, \langle \Sigma(z)\, \Sigma^\dagger(z')\rangle$$

and satisfies the equation

$$(\Box_{\rm v} - |\mathcal{M}_{\rm R}|^2)\, G^{(\rm R)}(z,z') = -\mathbf{1}\, \delta^8(z-z')~.$$

Important identities:

$$\mathcal{D}^\alpha \mathcal{W}_\alpha = 0 \quad \Longrightarrow$$

$$\Box_{\rm v}\, \bar{\mathcal{D}}^2 = \Box_{\rm v}\, \bar{\mathcal{D}}^2~, \qquad \Box_{\rm v}\, \mathcal{D}^2 = \Box_{\rm v}\, D^2~,$$

$$\Box_{\rm v}\, \bar{\mathcal{D}}^2 = \Box_{+}\, \bar{\mathcal{D}}^2~, \qquad \Box_{\rm v}\, \mathcal{D}^2 = \Box_{-}\, D^2~.$$

The Feynman propagators in the model $S^{(2)}_{\rm SYM} + S_{\rm GF}$ are

$$\begin{aligned}
\frac{\rm i}{g^2}\,\langle v(z)\, v^{\rm T}(z')\rangle &= -G^{(\rm Ad)}(z,z')\ , \\
\frac{\rm i}{g^2}\,\langle \varphi(z)\, \varphi^{\dagger}(z')\rangle &= \frac{1}{16}\bar{\mathcal{D}}^2\mathcal{D}'^2\, G^{(\rm Ad)}(z,z')\ , \\
\langle \varphi(z)\, \varphi^{\rm T}(z')\rangle &= \langle \bar{\varphi}(z)\, \varphi^{\dagger}(z')\rangle = 0\ .
\end{aligned}$$

It is understood here that v and φ are column-vectors, and not matrices as in the preceding consideration.

The Feynman propagators for the action $S^{(2)}_{\rm gh}$ are:

$$ {\rm i}\,\langle \tilde{c}(z)\, c^{\dagger}(z')\rangle = -{\rm i}\,\langle c(z)\, \tilde{c}^{\dagger}(z')\rangle = \frac{1}{16}\bar{\mathcal{D}}^2\mathcal{D}'^2\, G^{(\rm Ad)}(z,z')\ . $$

To formulate the Feynman propagators in the model $S^{(2)}_{\rm hyper}$, it is useful to introduce the notation

$$\mathbf{q} = \begin{pmatrix} q \\ \tilde{q}^{\rm T} \end{pmatrix} ~, \qquad \mathbf{q}^\dagger = (q^\dagger , \, \bar{\tilde{q}}) ~.$$

Then, the Feynman propagators read

$$\begin{aligned} {\rm i}\, \langle \mathbf{q}(z)\, \mathbf{q}^\dagger(z') \rangle &= \frac{1}{16} \bar{\mathcal{D}}^2 \mathcal{D}'^2 \, G^{({\rm R}\oplus{\rm R_c})}(z,z') ~, \\ {\rm i}\, \langle q(z)\, \tilde{q}(z') \rangle &= \mathcal{M}^\dagger_{\rm R} \, G^{({\rm R})}_+(z,z') ~, \\ {\rm i}\, \langle \tilde{q}^\dagger(z)\, q^\dagger(z') \rangle &= \mathcal{M}_{\rm R} \, G^{({\rm R})}_-(z,z') ~, \end{aligned}$$

where the covariantly chiral (G_+) and antichiral (G_-) Green's functions are related to G as follows:

$$\begin{aligned} G_+(z,z') &= -\frac{1}{4} \bar{\mathcal{D}}^2 G(z,z') = -\frac{1}{4} \bar{\mathcal{D}}'^2 G(z,z') ~, \\ G_-(z,z') &= -\frac{1}{4} \mathcal{D}^2 G(z,z') = -\frac{1}{4} \mathcal{D}'^2 G(z,z') ~. \end{aligned}$$

Exercise

Demonstrate that the cubic and quartic parts of $S_{\rm vect}$ are:

$$
\begin{aligned}
S^{(3)}_{\rm vect} &= \frac{1}{2}{\rm tr}\int {\rm d}^8z\,[v,\mathcal{D}^\alpha v]\,\left(\frac{1}{8}\,\bar{\mathcal{D}}^2\mathcal{D}_\alpha v+\frac{1}{3}[\mathcal{W}_\alpha,v]\right)~,\\
S^{(4)}_{\rm vect} &= -\frac{1}{8}{\rm tr}\int {\rm d}^8z\,[v,\mathcal{D}^\alpha v]\,\left(\frac{1}{8}\bar{\mathcal{D}}^2[v,\mathcal{D}_\alpha v]\right.\\
&\qquad\qquad\left.-\frac{1}{6}[v,\bar{\mathcal{D}}^2\mathcal{D}_\alpha v]+\frac{1}{3}[v,[v,\mathcal{W}_\alpha]]\right)~.
\end{aligned}
$$

Using the algebra of gauge-covariant derivatives that the functionals $S^{(3)}_{\rm vect}$ and $S^{(4)}_{\rm vect}$ are real modulo total derivatives.