Chapter 1

Michael Fisher and Localization

Franz J. Wegner

Institute for Theoretical Physics,
Philosophenweg 19,
Ruprecht-Karls-University,
D-69120 Heidelberg, Germany

Michael Fisher presented many important contributions to critical phenomena and distributed it widely in review articles. His paper with Kenneth Wilson on the expansion of critical exponents in $4-\epsilon$ dimensions gave the field of critical phenomena an enormous boost.
A few years later Alexander Polyakov presented his paper on the expansion around the lower critical dimension 2, which brought an important impetus for non-linear sigma-models. The Anderson localization transition due to disorder is described by such a model of supersymmetric matrices. It is applied to various dimensions and to ensembles with and without time-reversal invariance.

1. Critical phenomena and Michael Fisher

Michael Fisher was one of the leading scientists in the development of critical phenomena. Cyril Domb describes[1] the first 16 years (1948-1966) of Michael's academic career at King's College in London, where he started as an undergraduate and ended as full professor. He excelled by his mathematical abilities. Domb's inaugural lecture 1955 *Statistical Physics and its problems* directed Michael to critical phenomena and related problems, which he started with enumeration of graphs and paths on lattices.[2]

Michael Fisher worked on high- and low-temperature expansions for various vector models in statistical mechanics. These expansions allow to estimate critical temperatures and critical exponents. He mastered various systems connected with critical behaviour. He presented vividly his knowledge in lectures and review articles. I mention only the References,[3–6] which gave him the title *Pope of Critical Phenomena*.

In 1966 Michael went to the Physics Department at Cornell University. Probably his most important paper was that with Ken Wilson.[7] In this paper they expanded the critical exponents for the Ising- and the XY-model in an ϵ-expansion in $d=4-\epsilon$ dimensions, starting out from the recurrence relation derived by Wilson.[8]

This paper was seminal to many papers. Initially most of them were on the isotropic and anisotropic n-vector model, as can be seen for example in the review-articles in volume 6 of the series on *Phase Transitions and Critical Phenomena*,[9] the first volume of this series after the appearance of the Wilson-Fisher paper.

However, some problems, such as the localization problem, call for matrix models. This will be considered in the following Section.

2. Anderson Localization

2.1. *Introduction*

Phil Anderson argued in his 1958 paper[10] that disorder in a metal destroys diffusion close to the band tails, but it may still allow diffusion and conduction in the centre of the band. Thus at a certain Fermi-energy, called mobility edge, the system switches from insulating to conducting behaviour. The eigenstates in the insulating regime are localized, whereas they are extended in the conducting regime. David Thouless gives a historical survey[11] in the volume *50 years of Anderson localization*.[12] Interest in this phenomenon grew and attracted an increasing number of physicists, in particular Neville Mott[13,14] and coworkers and also David Thouless.[15] One of the problems was to distinguish, what is due to disorder and what is due to interactions like coupling to phonons or Coulomb interaction between electrons.

Here I consider only the localization problem due to disorder in the one-particle Hamiltonian as described by the Hamiltonian in Eq. (1). It became clear that the localization transition is a critical phenomenon.

One observation was that the amplitudes at the sites r occur pairwise. This became clear from the $1/n$-expansions of the corresponding n-orbital models (Oppermann and Wegner[16]), but also in the approach by Aharony and Imry,[17] and in the calculation by Harris and Lubensky,[18,19] although the later approaches did not come to valid conclusions. The $1/n$-expansion yields the leading terms of the critical exponents.

The transformation to the non-linear sigma-model was performed in the papers by Wegner[20] and by Schäfer and Wegner[21] in 1979 and 1980. There the replica trick was used: The calculation was performed for m replicas and finally the limit $m \to 0$ was taken. The result is a non-linear sigma-model of a matrix field Q, which obeys $Q^2 = 1$. The symmetry of Q is of hyperbolic nature. The correlations of the amplitudes of this model are expressed by correlations of elements of the Q-matrices. Efetov et al.[22] performed a similar calculation with the replica trick, however representing the amplitudes by fermionic variables. They obtained the renormalization in one-loop order for all three Dyson classes. The critical behaviour of the non-linear sigma-models were analyzed with the renormalization group procedure of Polyakov[23] starting from the lower critical dimension two.

In 1979 the paper by Abrahams, Anderson, Licciardello, and Ramakrishnan[24] appeared with their renormalization approach for the conductance.

The replica trick may sometimes fail, see Ref.[25] However, in 1982 Efetov[26] introduced Grassmann variables in addition to the complex variables and thus avoided the replica trick and gave safer ground by this super-symmetric formulation of the non-linear sigma-model.

There are several review articles on this non-linear sigma-model and related topics. I mention only those by Efetov,[27,28] Verbaarschott et al.,[29] Belitz et al.,[30] Zirnbauer,[31–33] Guhr et al.,[34] Mirlin,[35] Evers et al.,[36] Abrahams in *Fifty years of Anderson Localization*,[12] and chapters 4 and 21-23 of Wegner.[39]

Subsection 2.2 follows with preliminaries. The tight-binding model is described. Then I explain the three symmetry classes according to Dyson.[40] Finally a few relations for anticommuting variables and for supermatrices are given.

In subsection 2.3 the non-linear sigma model for the tight-binding model is developed including the Hubbard-Stratonovich transformation.

In subsection 2.4 on the random matrix model we extend the models to n-orbital models. Then the technique will be used for the zero-dimensional case, the problem originally introduced by Wigner[41] as description for the statistics of nuclear levels. The correlations between the energy levels are given for the three Dyson ensembles.

In subsection 2.5 on hydrodynamics diffusion and conductivity are considered in the limit of small wave-vectors q and frequencies ω. We emphasize diffusons and the importance of cooperons in case of time-reversal invariance. For the Dyson-ensembles symmetry relations and β-functions are given both in powers of the inverse diffusion constant and in the inverse conductance. Remarks on density fluctuations and multifractality and the statistics of energy spacing are added.

A few other developments are mentioned in the final paragraph: All compact symmetric spaces correspond to non-linear sigma-models which adds to the three Wigner-Dyson classes three chiral classes and four Bogolubov-de Gennes classes. Exact proof of absence of diffusion for large disorder or low energies is mentioned. Further numerically determined exponents are listed. A few authors who worked on interacting disordered systems are cited.

2.2. *Preliminaries*

Tight-binding model I start out with the diffusive model, which is a tight-binding model with orbitals $|r\rangle$ at sites r and hopping matrix elements $f_{r,r'} + t_{r,r'}$

$$H = \sum_{r,r'} |r\rangle (f_{r,r'} + t_{r,r'}) \langle r'| \tag{1}$$

The elements $f_{r,r'}$ are random matrix elements with average zero. For the sake of simplicity they are assumed to be Gaussian distributed

$$\overline{f_{r,r'}} = 0, \quad \overline{f_{r_1,r_1'} f_{r_2,r_2'}} = M_{r_1,r_1'} \delta_{r_1,r_2'} \delta_{r_2,r_1'} \tag{2}$$

$M_{r,r'}$ is assumed to be translational invariant,

$$M_{r,r'} = M_{r-r'} \tag{3}$$

In contrast t is assumed to be fixed, but also translational invariant,

$$t_{r,r'} = t_{r-r'}. \tag{4}$$

I will also consider the n-orbital model, where there are n orbitals at each lattice site.

Time-reversal invariance In accord with Dyson[40] we consider ensembles with three symmetries. The unitary ensemble applies to systems without time-reversal invariance. For $t_{r-r'} = 0$ it is also called phase-invariant ensemble, since it is invariant against transformations $|r\rangle \to \exp(\mathrm{i}\phi_r)|r\rangle$.

Disordered systems with time-reversal invariance are described by orthogonal and symplectic ensembles, where the first one neglects spin, and the second one takes spin-dependent hopping into account. The orthogonal ensemble is also called real-matrix ensemble, since the matrix elements f and t are real. The symplectic ensemble takes spin 1/2 into account. To preserve time-reversal invariance the matrix elements are real quaternions. This is described in Eqs. (108-111).

Grassmann variables Grassmann variables are anticommuting variables. Thus

$$\xi_1\xi_2 = -\xi_2\xi_1, \quad \xi_1^2 = 0 \tag{5}$$

for Grassmann variables ξ_1, ξ_2. Products of an even (odd) number of Grassmann variables and sums of these products are called even (odd) elements of the algebra.

These variables were introduced by Grassmann and later independently by Berezin.[42,43] For the algebra, analysis, and groups of these variables see Refs.[27,28,39,43,44]

Their usefulness becomes obvious from the Gaussian integrals

$$\int \exp(-\sum x_i^* A_{ij} x_j) \prod_i \frac{\mathrm{d}\Re x_i \mathrm{d}\Im x_i}{\pi} = (\det A)^{-1}, \tag{6}$$

$$\int \exp(-\sum \xi_i^* A_{ij} \xi_j) \prod_i \mathrm{d}\xi_i^* \mathrm{d}\xi_i = \det A \tag{7}$$

for complex x_i and Grassmannian ξ_i. The integrations in the first integral run from $-\infty$ to $+\infty$. The real parts of the eigenvalues of A have to be positive. The ξ and ξ^* have to be linearly independent in the second integral.

Supermatrices Let us denote the set of $(n+m)\times(n+m)$-matrices

$$K = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \tag{8}$$

by $\mathcal{M}(n,m)$, where a and α have n rows, β and b m rows, a and β n columns, and α and b have m columns. The matrices a and b consist of even elements, and α and β of odd elememts. a is called boson-boson matrix, b fermion-fermion matrix. For these matrices one defines the supertrace

$$\mathrm{str}(K) = \mathrm{tr}(a) - \mathrm{tr}(b) \tag{9}$$

It obeys for matrices $K, L \in \mathcal{M}(n,m)$

$$\mathrm{str}(KL) = \mathrm{str}(LK). \tag{10}$$

The superdeterminant is given by

$$\mathrm{sdet}(K) = \frac{\det(a)}{\det(b - \beta a^{-1}\alpha)} = \frac{\det(a - \alpha b^{-1}\beta)}{\det(b)}. \tag{11}$$

It obeys

$$\mathrm{sdet}(KL) = \mathrm{sdet}(K)\,\mathrm{sdet}(L). \tag{12}$$

One can express the superdeterminant by the supertrace

$$\mathrm{sdet}(K) = \exp(\mathrm{str}(\ln K)) \tag{13}$$

and its differential as

$$\mathrm{d}\,\mathrm{sdet}(K) = \mathrm{sdet}(K)\,\mathrm{str}(K^{-1}\mathrm{d}K). \tag{14}$$

2.3. *Tight-binding model*

Green's functions The one-particle Green's functions

$$G(r, r', z) = \langle r|\frac{1}{z - H}|r'\rangle \tag{15}$$

are obtained from integrals over time t

$$\mathrm{i}\int_{-\infty}^{0} \mathrm{d}t \mathrm{e}^{(\mathrm{i}E+\eta-\mathrm{i}H)t} = \frac{1}{E - \mathrm{i}\eta - H} \quad \Im z < 0, \tag{16}$$

$$-\mathrm{i}\int_{0}^{+\infty} \mathrm{d}t \mathrm{e}^{(\mathrm{i}E-\eta-\mathrm{i}H)t} = \frac{1}{E + \mathrm{i}\eta - H} \quad \Im z > 0. \tag{17}$$

The upper Green's function is called advanced, the lower one retarded. The density of states ρ at site r is obtained as the limit

$$\rho(r, E) = \lim_{\eta\to+0}(G(r, r, E - \mathrm{i}\eta) - G(r, r, E + \mathrm{i}\eta))/(2\pi\mathrm{i}). \tag{18}$$

The product of m Green's functions for the tight-binding Hamiltonian (1) reads

$$\begin{aligned}\prod_{p=1}^{m} G(r_p, r'_p, z_p) = &\int \prod_{p=1}^{m} s_{p\mathrm{b}} x_p(r_p) x^*_p(r'_p) \\ &\times \exp(-\sum_{p,r,r'}[s_{p\mathrm{b}} x^*_p(r)(z_p\delta_{r,r'} - f_{r,r'} - t_{r.r'})x_p(r') \\ &+ s_{p\mathrm{f}}\xi^*_p(r)(z_p\delta_{r,r'} - f_{r,r'} - t_{r,r'})\xi_p(r)]) \prod_{r,p}(\frac{s_{p\mathrm{f}}}{s_{p\mathrm{b}}}\mathrm{D}[S_p(r)]),\end{aligned} \tag{19}$$

with the integrals over the complex variables x and the Grassmann variables ξ

$$(S^*_p(r) = (x^*_p(r), \xi^*_p(r)), \quad S_p(r) = \begin{pmatrix} x_p(r) \\ \xi_p(r) \end{pmatrix}, \tag{20}$$

$$\mathrm{D}[S_p(r)] = \int \frac{\mathrm{d}\Re x_p(r)\mathrm{d}\Im x_p(r)}{\pi}\mathrm{d}\xi^*_p(r)\mathrm{d}\xi_p(r). \tag{21}$$

Convergence requires the choice $s_{p\mathrm{b}} = -\mathrm{i}\Im z_p$ for the integration of x_p. We introduce

$$A_p(r,r') = s_{p\mathrm{b}} x_p^*(r) x_p(r') + s_{p\mathrm{f}} \xi_p^*(r) \xi_p(r') \tag{22}$$

and the functions

$$\mathcal{S}_{\mathrm{int}} = \sum_{p,r,r'} A_p(r,r')(f_{r,r'} + t_{r,r'}), \tag{23}$$

$$\begin{aligned}\mathcal{S}_{z,J} = &\sum_{p,r} s_{\mathrm{b}p} z_p x_p^*(r) x_p(r) + \sum_{p,r} s_{\mathrm{f}p} z_p \xi_p^*(r) \xi_p(r) \\ &- \sum_{p,p',r,r'} \sqrt{s_{p\mathrm{b}} s_{p'\mathrm{b}}} J_{p',p}(r',r) x_{p'}^*(r') x_p(r).\end{aligned} \tag{24}$$

The contributions J are source terms. They allow the determination of Green's functions and products of Green's functions. The partition function

$$Z(J) = \int \exp(-\mathcal{S}_{\mathrm{int}} - \mathcal{S}_{z,J}) \prod_{p,r} \mathrm{D}[S_p(r)] \tag{25}$$

yields

$$\frac{\partial Z(J)}{Z(0)\partial J_{p,p}(r',r)} = G(r,r',z_p), \tag{26}$$

$$\frac{\partial^2 Z(J)}{Z(0)\partial J_{p_1,p_1}(r_1',r_1)\partial J_{p_2,p_2}(r_2',r_2)} = G(r_1,r_1',z_{p_1})G(r_2,r_2',z_{p_2}), \tag{27}$$

$$\frac{\partial^2 Z(J)}{Z(0)\partial J_{p_1,p_2}(r_1',r_2)\partial J_{p_2,p_1}(r_2',r_1)} = G(r_1,r_1',z_{p_1})G(r_2,r_2',z_{p_2}). \tag{28}$$

Ensemble average and Hubbard-Stratonovich transformation We separate

$$\mathcal{S}_{\mathrm{int}} = \mathcal{S}_{\mathrm{fluc}} + \mathcal{S}_{\mathrm{t}}, \quad \mathcal{S}_{\mathrm{fluc}} = \sum_{p,r,r} f_{r,r'} A_p(r,r'), \quad \mathcal{S}_{\mathrm{t}} = \sum_{p,r,r'} t_{r.r'} A_p(r,r'). \tag{29}$$

The Gaussian distribution (2) of f yields the ensemble average

$$\overline{\exp(-\mathcal{S}_{\mathrm{fluc}})} = \exp(\frac{1}{2} \sum_{p,p',r,r'} M_{r,r'} A_p(r,r') A_{p'}(r',r)) \tag{30}$$

The product $A_p A_{p'}$ contains products of four spin variables. They can be eliminated by means of the Hubbard-Stratonovich transformation.

For this purpose we introduce the inverse w of M,

$$(M^{-1})_{r,r'} = w_{r,r'}, \tag{31}$$

In order that the matrices M and w are positive definite, we require $w_{r,r'} \leq 0$ for $r \neq r'$. From now on we restrict ourselves to $m = 2$ energies z_1, z_2 and Green's functions and put them together

$$S(r) = \begin{pmatrix} x_{p_1}(r) \\ x_{p_2}(r) \\ \xi_{p_1}(r) \\ \xi_{p_2}(r) \end{pmatrix}. \tag{32}$$

and use

$$s = \mathrm{diag}(s_{p_1\mathrm{b}}, s_{p_2\mathrm{b}}, s_{p_1\mathrm{f}}, s_{p_2\mathrm{f}}), \tag{33}$$

$$z = \mathrm{diag}(z_{p_1}, z_{p_2}, z_{p_1}, z_{p_2}). \tag{34}$$

Then

$$A(r, r') = S^*(r) s S(r') \tag{35}$$

and

$$A(r, r') A(r', r) = \mathrm{str}(B(r) B(r')) \tag{36}$$

holds with

$$B(r) = \sqrt{s} S(r) S^*(r) \sqrt{s} \in \mathcal{M}(2,2). \tag{37}$$

Hence we have

$$\overline{\exp(-\mathcal{S}_{\mathrm{fluc}})} = \exp(\frac{1}{2} \sum_{r,r'} M_{r,r'} \,\mathrm{str}(B(r) B(r'))). \tag{38}$$

We introduce fields $R(r) \in \mathcal{M}(2,2)$, where two Green's functions are considered for energies $E \pm \omega/2$. Then also B belongs to $\mathcal{M}(2,2)$.

We observe

$$\begin{aligned} \sum_{r,r'} w_{r,r'} \quad & \mathrm{str}[(R(r) - \sum_{r'_1} M_{r,r'_1} B(r'_1)) \\ & \times (R(r') - \sum_{r_1} M_{r',r_1} B(r_1))] \\ & = \sum_{r,r'} w_{r,r'} \,\mathrm{str}(R(r) R(r')) - \sum_r \mathrm{str}(R(r) B(r)) \\ & \quad - \sum_{r'} \mathrm{str}(B(r') R(r')) + \sum_{r,r'} M_{r',r} \,\mathrm{str}(B(r') B(r)). \end{aligned} \tag{39}$$

One multiplies the expression (39) by $-1/2$, exponentiates it and integrates it over R. This integral yields one. One multiplies the integral by the expression for the average of the fluctuation average (38). Then one is left with

$$\begin{aligned} \overline{\exp(-\mathcal{S}_{\mathrm{fluc}})} = \prod_r \int \mathrm{D}[R(r)] \exp(-\frac{1}{2} \sum_{r,r'} w_{r,r'} \,\mathrm{str}(R(r) R(r')) \\ + \sum_r \mathrm{str}(B(r) R(r))). \end{aligned} \tag{40}$$

Eqs. (39) and (40) constitute the Hubbard-Stratonovich transformation.[37,38] Since

$$\mathrm{str}(B(r) R(r))) = \sum_r \mathrm{str}(S^*(r) \sqrt{s} R(r) \sqrt{s} S(r)) \tag{41}$$

we obtain

$$\begin{aligned} Z(J) = \int \prod_r (\frac{s_{p_1\mathrm{f}} s_{p_2\mathrm{f}}}{s_{p_1\mathrm{b}} s_{p_2\mathrm{b}}} \mathrm{D}[S_(r)]) \prod_r \mathrm{D}[R(r)] \exp(\mathrm{str}[-\frac{1}{2} \sum_{r,r'} w_{r,r'} R(r) R(r')] \\ + \sum_{r,r'} \mathrm{str}[S^*(r) \sqrt{s} (-z \delta_{r,r'} + t_{r,r'} + J(r,r') + R(r) \delta_{r,r'}) \sqrt{s} S(r')]). \end{aligned} \tag{42}$$

Since the vectors S appear only bilinearly, integration over S can be performed and yields

$$Z(J) = \int \prod_r \mathrm{D}[R(r)] \exp(\mathrm{str}[-\frac{1}{2}\sum_{r,r'} w_{r,r'} R(r)R(r')] \\ \times \mathrm{sdet}(R - z + t + J)^{-1}. \tag{43}$$

Fourier transform If the hopping matrix elements t vanish, then we can immediately go to the next step and determine the saddle point of the integrand of Eq. (43). However, with the hopping matrix elements t we have to perform a Fourier transform,

$$S(r) = \frac{1}{\sqrt{N}} \sum_q \mathrm{e}^{\mathrm{i}qr} \hat{S}(q), \quad S^*(r) = \frac{1}{\sqrt{N}} \sum_q \mathrm{e}^{-\mathrm{i}qr} \hat{S}^*(q). \tag{44}$$

and

$$\hat{t}_q = \frac{1}{N} \sum_{r,r'} \mathrm{e}^{\mathrm{i}q(r'-r)} t_{r,r'}, \quad \mathcal{S}_\mathrm{t} = \sum_{p,q} \hat{t}_q \hat{S}^*(q) \hat{S}(q) \tag{45}$$

with q confined to the first Brillouin zone. N is the number of lattice sites. We assume periodic boundary conditions. We indicate the Fourier transformed quantities by hats.

The corresponding transformation for (31) yields for $\hat{M}$ and $\hat{w}$

$$\hat{M}_q \hat{w}_q = 1. \tag{46}$$

The choice

$$\hat{R}_{q-q'} = \frac{1}{N} \sum_r \mathrm{e}^{\mathrm{i}(q'-q)r} R(r). \tag{47}$$

yields

$$\sum_{q,q'} \hat{S}^*(q) \sqrt{s} \hat{R}_{q-q'} \sqrt{s} \hat{S}(q') = \sum_r S^*(r) \sqrt{s} R(r) \sqrt{s} S(r) \tag{48}$$

for the expression (41).

Saddle point One obtains the saddle point of the integrand of $Z(0)$, Eq. (43) by setting $R(r) = R^{(0)}$ for all r. Then the saddle point equation reads

$$\hat{w}_0 R^{(0)} + \frac{1}{N} \sum_q \frac{1}{R^{(0)} - z + \hat{t}_q} = 0. \tag{49}$$

There are $N-1$ solutions along the real axis between the lowest and largest values of $z_p - \hat{t}_q$. Besides there is a pair of solutions outside the real axis, if z is inside the band. They are relevant for the saddle point and diagonal in p and ν,

$$R^{(0)}_{p\nu,p'\nu'}(r) = \delta_{p,p'} \delta_{\nu,\nu'} R^\mathrm{d}_p. \tag{50}$$

The one-particle Green's function is given within the saddle point approximation

$$\hat{G}^{(0)}(q, z_p) = \frac{1}{z_p - R^\mathrm{d}_p - \hat{t}_q}. \tag{51}$$

Second order and fluctuations Returning to Z in Eq. (43) we expand

$$\mathrm{sdet}(R-z+t+J)=\mathrm{sdet}(R-z+t)\,\mathrm{sdet}(1-\mathcal{G}J),\quad \mathcal{G}:=\frac{1}{z-R-t} \tag{52}$$

Then we obtain with $R=R^{(0)}+\delta R$

$$\mathcal{G}=G^{(0)}+G^{(0)}\delta RG^{(0)}+G^{(0)}\delta RG^{(0)}\delta RG^{(0)}+... \tag{53}$$

and the one-particle Green's function up to order $(\delta R)^2$

$$\hat{G}_1(q,z_p)=\hat{G}^{(0)}(q,z_p)+(\hat{G}^{(0)}(q,z_p))^2\sum_{q',p'\nu}\hat{G}^{(0)}(q',z_{p'})\overline{\delta\hat{R}_{q-q',p\mathrm{b},p'\nu}\delta\hat{R}_{q'-q,p'\nu,p\mathrm{b}}}, \tag{54}$$

where $\nu=0$ stands for b and $\nu=1$ for f.

The two-particle function yields

$$\begin{aligned}\hat{G}_2(q_1,q_1',z_1;q_2,q_2',z_2)&=\overline{\hat{\mathcal{G}}_{1\mathrm{b},1\mathrm{b}}(q_1,q_1')\hat{\mathcal{G}}_{2\mathrm{b},2\mathrm{b}}(q_2,q_2')}+\overline{\hat{\mathcal{G}}_{1\mathrm{b},2\mathrm{b}}(q_1,q_2')\hat{\mathcal{G}}_{2\mathrm{b},1\mathrm{b}}(q_2,q_1')}\\&=\delta_{q_1,q_1'}\hat{G}_1(q_1,z_1)\delta_{q_2,q_2'}\hat{G}_1(q_2,z_2)\\&+\hat{G}_1(q_1,z_1)\hat{G}_1(q_1',z_1)\hat{G}_1(q_2,z_2)\hat{G}_1(q_2',z_2)\\&\times(\overline{\delta\hat{R}_{q_1-q_1',1\mathrm{b},1\mathrm{b}}\delta\hat{R}_{q_2-q_2',2\mathrm{b},2\mathrm{b}}}+\overline{\delta\hat{R}_{q_1-q_2',1\mathrm{b},2\mathrm{b}}\delta\hat{R}_{q_2-q_1',2\mathrm{b},1\mathrm{b}}}).\end{aligned} \tag{55}$$

The action up to second order in δR reads

$$\begin{aligned}\mathcal{S}(R,0)&=\mathcal{S}(R^{(0)},0)+\frac{1}{2}\sum_{r,r'}\mathrm{str}(\delta R(r)w_{r,r'}\delta R(r'))-\frac{1}{2}\mathrm{str}(G^{(0)}\delta RG^{(0)}\delta R)\\&=\frac{1}{2}N\sum_{q,p,\nu,p',\nu'}(-)^\nu(\hat{w}_q-\hat{\Pi}_q(z_{p'},z_p))\delta\hat{R}_{-q,p\nu,p'\nu'}\delta\hat{R}_{q,p'\nu',p\nu}.\end{aligned} \tag{56}$$

with

$$\hat{\Pi}_q(z_{p'},z_p)=\sum_{q'}\hat{G}^{(0)}(q+q',z_p)\hat{G}^{(0)}(q',z_{p'}). \tag{57}$$

This is the action in harmonic approximation in δR. It yields

$$\overline{\delta\hat{R}_{-q_1,p_1\nu_1,p_1',\nu_1'}\delta\hat{R}_{q_2,p_2'\nu_2',p_2\nu_2}}=\delta_{q_1,q_2}\delta_{p_1,p_2}\delta_{p_1',p_2'}\delta_{\nu_1,\nu_2}\delta_{\nu_1',\nu_2'}\frac{(-)^{\nu_1'}}{N}\hat{\Gamma}_{q_1}(z_{p_1'},z_{p_1}) \tag{58}$$

with

$$\hat{\Gamma}_q(z_{p'},z_p)=\frac{1}{\hat{w}_q-\hat{\Pi}_q(z_{p'},z_p)}. \tag{59}$$

The two contributions $\nu=0$ and $\nu=1$ for the one-particle Green's function $\hat{G}_1$ cancel. Only the second fluctuation term to $\hat{G}_2$ contributes,

$$\begin{aligned}&\hat{G}_2(q_1,q_1',z_1;q_2,q_2',z_2)=\delta_{q_1,q_1'}\hat{G}_1(q_1,z_1)\delta_{q_2,q_2'}\hat{G}_1(q_2,z_2)\\&+\frac{\delta_{q_1+q_2,q_1'+q_2'}}{N}\hat{\Gamma}_{q_1-q_1'}(z_1,z_2)\hat{G}_1^{(0)}(q_1,z_1)\hat{G}_1^{(0)}(q_1',z_1)\hat{G}_1^{(0)}(q_2,z_2)\hat{G}_1^{(0)}(q_2',z_2).\end{aligned} \tag{60}$$

Nonlinear σ-model We choose now $z = E \pm \mathrm{i}0$ and obtain

$$R^{(0)}(E) = \mathrm{diag}(R_{\mathrm{A}}(E), R_{\mathrm{R}}(E), R_{\mathrm{A}}(E), R_{\mathrm{R}}(E)). \tag{61}$$

The density of states is given by Eq. (18). Thus

$$2\pi\rho = \hat{w}_0(R_{\mathrm{A}}(E) - R_{\mathrm{R}}(E)) \tag{62}$$

$$R = \Re R^{(0)}\mathbf{1}_4 + \frac{\pi \mathrm{i}\rho}{\hat{w}_0}\Lambda \tag{63}$$

with

$$\Lambda = \mathrm{diag}(1, -1, 1, -1). \tag{64}$$

Eq. (61) is not the general solution for the saddle point. This is given by

$$R(E,T) = \Re R^{(0)}\mathbf{1}_4 + \frac{\pi \mathrm{i}\rho}{\hat{w}_0}Q(T) \tag{65}$$

with

$$Q^2(T) = \mathbf{1}_4. \tag{66}$$

Then the saddle point Eq. (49) reads

$$\hat{w}_0\Re R^{(0)} + \pi \mathrm{i}\rho Q = \frac{1}{N}\sum_q \frac{E - \Re R^{(0)} - \hat{t}_q + \pi \mathrm{i}\rho Q/\hat{w}_0}{(E - \Re R^{(0)} - \hat{t}_q)^2 + (\pi\rho/\hat{w}_0)^2}. \tag{67}$$

The full matrix R is

$$R = \Re R^{(0)}\mathbf{1}_4 + (\pi \mathrm{i}\rho/\hat{w}_0)Q + P \tag{68}$$

The symmetry of R is the same as of B, since both are multiplied by R in Eq. (40),

$$B^\dagger = s^{-1}Bs. \tag{69}$$

thus

$$Q^\dagger(T) = s^{-1}Q(T)s. \tag{70}$$

$$Q(T) = T\Lambda T^{-1}, \quad TsT^\dagger = s, \tag{71}$$

Q can be written as suggested by Efetov[28]

$$Q = U \begin{pmatrix} \lambda_1 & \mathrm{i}\mu_1 & 0 & 0 \\ \mathrm{i}\mu_1^* & -\lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & \mu_2 \\ 0 & 0 & \mu_2^* & -\lambda_2 \end{pmatrix} U^{-1} \tag{72}$$

with

$$\lambda_1 = \cosh\theta_1, \;\; \mu_1 = \sinh\theta_1 \exp(\mathrm{i}\phi_1), \tag{73}$$

$$\lambda_2 = \cos\theta_2, \;\; \mu_2 = \sin\theta_2 \exp(\mathrm{i}\phi_2). \tag{74}$$

The matrix T has the form of Q, but θ_i is replaced by $\theta_i/2$. The matrix U differs from the unit matrix only by anticommuting terms,

$$U^{\pm 1} = \begin{pmatrix} 1-\frac{1}{2}\alpha^*\alpha & 0 & \mp\alpha^* & 0 \\ 0 & 1+\frac{1}{2}\beta^*\beta & 0 & \mp\mathrm{i}\beta^* \\ \pm\alpha & 0 & 1-\frac{1}{2}\alpha\alpha^* & 0 \\ 0 & \pm\mathrm{i}\beta & 0 & 1+\frac{1}{2}\beta\beta^* \end{pmatrix}. \tag{75}$$

There exist several parametrizations for the matrix P. We give here the Schäfer-Wegner parametrization

$$P(r) = T_0 \begin{pmatrix} P^{\mathrm{A}}_{\mathrm{bb}}(r) & 0 & P^{\mathrm{A}}_{\mathrm{bf}}(r) & 0 \\ 0 & P^{\mathrm{R}}_{\mathrm{bb}}(r) & 0 & P^{\mathrm{R}}_{\mathrm{bf}}(r) \\ P^{\mathrm{A}}_{\mathrm{fb}}(r) & 0 & P^{\mathrm{A}}_{\mathrm{ff}}(r) & 0 \\ 0 & P^{\mathrm{R}}_{\mathrm{fb}}(r) & 0 & P^{\mathrm{R}}_{\mathrm{ff}}(r) \end{pmatrix} T_0^{-1} \tag{76}$$

with fixed T_0, which is of the form T. P_{bb} is hermitian and P_{ff} antihermitian.

Both energies advanced or retarted If in contrast to eq. (61) both energies z_p are advanced or retarted, then we may use that

$$K(z+\frac{\omega}{2}, z-\frac{\omega}{2}) = K(z,z) - \frac{\omega^2}{4}\frac{\mathrm{d}^2 K(z_1,z_2)}{\mathrm{d}z_1\mathrm{d}z_2}|_{z_1=z_2=z} + ... \tag{77}$$

Thus only corrections in order ω^2 appear, which can be neglected for small ω.

Invariant measure One obtains for the invariant measure in the space of the Q-matrix

$$[\mathrm{D}Q] = \frac{1}{(\lambda_1-\lambda_2)^2}\mathrm{d}\lambda_1\mathrm{d}\lambda_2\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}\alpha\mathrm{d}\alpha^*\mathrm{d}\beta\mathrm{d}\beta^* \tag{78}$$

In order to obtain convergence we had to choose different signs s_{Ab} and s_{Rb} for the bosonic integrations. We choose for the fermionic integrations $s_{\mathrm{Af}} = s_{\mathrm{Rf}}$. Then the singularity $1/(\lambda_1-\lambda_2)^2$ appears only at $\lambda_1 = \lambda_2$.

Then the λ-integrals run over the intervals $\lambda_1 = 1...\infty$ and $\lambda_2 = -1...+1$. Closer investigation shows that the singularity at $\lambda_1 = \lambda_2 = 1$ creates a contribution at these λs with $\alpha = \alpha^* = \beta = \beta^* = 0$.

2.4. *The Random Matrix Model*

***n*-Orbital models** Investigation of the n-orbital model in the limit $n \to \infty$[16,45] allowed the determination of critical exponents at the mobility edge up to order $1/n^2$.

Compared to the model (1) an additional index indicated by a Greek letter running from 1 to n is introduced. Thus Eq. (1) is replaced by

$$H = \sum_{r\alpha, r'\beta} |r,\alpha\rangle(f_{r\alpha,r'\beta} + t_{r,r'}\delta_{\alpha,\beta})\langle r',\beta| \tag{79}$$

with Gaussian distributed elements f with average 0

$$\overline{f_{r\alpha,r'\alpha'}}=0,\quad \overline{f_{r_1\alpha_1,r_1'\alpha'}f_{r_2\alpha_2,r_2'\alpha_2'}}=\frac{1}{n}M_{r-r'}\delta_{r_1,r_2'}\delta_{\alpha_1,\alpha_2'}\delta_{r_2,r_1'}\delta_{\alpha_2,\alpha_1'}. \tag{80}$$

One uses

$$A(r\alpha,r'\beta)=S^*(r\alpha)sS(r'\beta),\quad B(r)=\sum_\alpha\sqrt{s}S(r\alpha)S^*(r,\alpha)\sqrt{s} \tag{81}$$

and one has to replace M by M/n and w by nw. Finally one obtains

$$\begin{aligned}Z(J)=(\int\prod_r\mathrm{D}[R(r)])\exp(\mathrm{str}[-\frac{n}{2}\sum_{r,r'}w_{r,r'}R(r)R(r')]\\ \times\,\mathrm{sdet}(R-z+t+J)^{-n}.\end{aligned} \tag{82}$$

instead of Eq. (43). Thus the action $\mathcal{S}$ is multiplied by n.

Random Matrix Theory. $d=0$ The random matrix model has been considered by Wigner[41] as a zero-dimensional model. It is the model Eq. (79), but with only one site r and matrix $t=0$. Thus the Hamiltonian reads

$$H=\sum_{\alpha,\beta}|\alpha\rangle f_{\alpha,\beta}\langle\beta|. \tag{83}$$

The elements f are Gaussian distributed with average 0 and width

$$\overline{f_{\alpha,\alpha'}}=0,\quad \overline{f_{\alpha_1,\alpha_1'}f_{\alpha_2,\alpha_2'}}=\frac{1}{n}M\delta_{\alpha_1,\alpha_2'}\delta_{\alpha_2,\alpha_1'}. \tag{84}$$

Initially Wigner considered the orthogonal case (see below). R obeys for the unitary case

$$wR^{(0)}+\frac{1}{R^{(0)}-E}=0 \tag{85}$$

with the solution

$$R^{(0)}=\frac{E}{2}\pm\mathrm{i}\sqrt{M-E^2/4},\quad \rho(E)=\frac{w}{\pi}\sqrt{M-E^2/4}. \tag{86}$$

This is the well-known semi-circle law for the density of states, which also holds in the orthogonal and the symplectic case. The density ρ is so normalized that its integral over E yields unity.

The one- and two-particle Green's functions G and K obey

$$\overline{G(\alpha,\beta,z)}=\delta_{\alpha,\beta}G(z), \tag{87}$$

$$\overline{G(\alpha,\beta,z_1)G(\gamma,\delta,z_2)}=\delta_{\alpha,\beta}\delta_{\gamma,\delta}K(z_1,z_2)+\delta_{\alpha,\delta}\delta_{\beta,\gamma}K'(z_1,z_2), \tag{88}$$

$$n^2\tilde{K}(z_1,z_2)=\sum_{\alpha,\beta}\overline{G(\alpha,\alpha,z_1)G(\beta,\beta,z_2)}=n^2K(z_1,z_2)+nK'(z_1,z_2), \tag{89}$$

$$n\tilde{K}'(z_1,z_2)=\sum_{\alpha,\beta}\overline{G(\alpha,\beta,z_1)G(\beta,\alpha,z_2)}, \tag{90}$$

$$\tilde{K}'(z_1,z_2)=K(z_1,z_2)+nK'(z_1,z_2)=\frac{G(z_2)-G(z_1)}{z_1-z_2}. \tag{91}$$

The matrix elements of $Q_{\rm bb}$, which are used to calculate the expectation values, see Eq. (72), read

$$Q_{\rm bb}^{\rm RR} = \lambda_1 + \alpha^*\alpha(\lambda_2 - \lambda_1), \tag{92}$$
$$Q_{\rm bb}^{\rm AA} = -\lambda_1 + \beta^*\beta(\lambda_2 - \lambda_1), \tag{93}$$
$$Q_{\rm bb}^{\rm RA} = {\rm i}\mu_1(1 - \frac{1}{2}\alpha^*\alpha)(1 + \frac{1}{2}\beta^*\beta) + {\rm i}\mu_2^*\alpha^*\beta, \tag{94}$$
$$Q_{\rm bb}^{\rm AR} = {\rm i}\mu_1^*(1 - \frac{1}{2}\alpha^*\alpha)(1 + \frac{1}{2}\beta^*\beta) + {\rm i}\mu_2\beta^*\alpha. \tag{95}$$

Thus

$$G(z_1) = w(\frac{z_1}{2} - {\rm i}\sqrt{M - \frac{z_1^2}{4}}), \;\; G(z_2) = w(\frac{z_2}{2} + {\rm i}\sqrt{M - \frac{z_2^2}{4}}). \tag{96}$$
$$\tilde{K}(z_1, z_1) = G^2(z_1), \;\; \tilde{K}(z_2, z_2) = G^2(z_2) \tag{97}$$

and

$$\tilde{K}(z_1, z_2) = G(z_1)G(z_2) + (\pi\rho)^2 \int_1^\infty {\rm d}\lambda_1 \int_{-1}^1 {\rm d}\lambda_2 \exp^{{\rm i}\pi f(\lambda_1 - \lambda_2)}, \tag{98}$$
$$\tilde{K}'(z_1, z_2) = (\pi\rho)^2 \int_1^\infty {\rm d}\lambda_1 \int_{-1}^1 {\rm d}\lambda_2 \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} \exp^{{\rm i}\pi f(\lambda_1 - \lambda_2)}. \tag{99}$$

The mean level spacing is $1/(n\rho)$. Thus

$$f = n\rho\omega. \tag{100}$$

h Hence

$$\tilde{K}(z_1, z_2) = G(z_1)G(z_2) + (\pi\rho)^2 \frac{2{\rm i}\sin(\pi f)}{(\pi f)^2} {\rm e}^{{\rm i}\pi f}, \tag{101}$$
$$\tilde{K}'(z_1, z_2) = (\pi\rho)^2 \frac{2{\rm i}}{\pi f}. \tag{102}$$

This allows us to calculate the correlations between the levels

$$\overline{\rho(E + \frac{\omega}{2})\rho(E - \frac{\omega}{2})} = \rho^2(E)C_{\rm U}(f), \tag{103}$$
$$C_{\rm U}(f) = 1 + \delta(f) - \frac{\sin^2(\pi f)}{f^2} = \delta(f) + \frac{(\pi f)^2}{3} + O(f^4). \tag{104}$$

The delta function describes the self-correlation. The self-repulsion yields the f^2 increase. Erdös[46] reports a general form for the level correlation of k levels.

Orthogonal Gaussian ensemble The matrix elements in this ensemble are real $f_{\alpha,\beta} = f_{\beta,\alpha} \in \mathbb{R}$ and Gaussian distributed with

$$\overline{f_{\alpha,\beta}} = 0, \quad \overline{f_{\alpha_1,\alpha_1'} f_{\alpha_2,\alpha_2'}} = \frac{1}{n} M(\delta_{\alpha_1,\alpha_2'}\delta_{\alpha_2,\alpha_1'} + \delta_{\alpha_1,\alpha_2}\delta_{\alpha_1',\alpha_2'}). \tag{105}$$

This model was introduced by Wigner as a stochastic model for nuclear levels.[47] The number of field components S and consequently also of R and Q have to be doubled. (The same applies for the symplectic case.)

One obtains for the level correlation

$$C_{\mathrm{O}}(f) = 1 + \delta(f) - \frac{\sin^2(\pi f)}{(\pi f)^2} - \left(\frac{\pi}{2}\mathrm{sign}(f) - \mathrm{Si}(\pi f)\right)\left(\frac{\cos(\pi f)}{\pi f} - \frac{\sin(\pi f)}{(\pi f)^2}\right)$$
$$= \delta(f) + \frac{\pi^2}{6}|f| + O(f^2). \tag{106}$$

with the sine integral

$$\mathrm{Si}(x) = \int_0^x \mathrm{d}y \frac{\sin(y)}{y}, \quad \lim_{x\to\infty} \mathrm{Si}(x) = \frac{\pi}{2}. \tag{107}$$

The self-repulsion yields an increase proportional to $|f|$.

Symplectic Gaussian ensemble Spin-dependent time-reversal invariant ensemble

$$H = \sum_{\alpha,m,\beta,m'} |\alpha, m\rangle f_{\alpha,m,\beta,m'} \langle \beta, m'| \tag{108}$$

Time-reversal invariance of H requires $\Theta H = H\Theta$ with the time-reversal operator $\Theta = -\mathrm{i}\sigma^y K$, where K denotes complex conjugation in real space. Thus the matrices f are real quaternions,

$$\begin{pmatrix} f_{\uparrow\uparrow} & f_{\uparrow\downarrow} \\ f_{\downarrow\uparrow} & f_{\downarrow\downarrow} \end{pmatrix} = \epsilon \begin{pmatrix} f^*_{\uparrow\uparrow} & f^*_{\uparrow\downarrow} \\ f^*_{\downarrow\uparrow} & f^*_{\downarrow\downarrow} \end{pmatrix} \epsilon^{-1} = \begin{pmatrix} f^*_{\downarrow\downarrow} & -f^*_{\downarrow\uparrow} \\ -f^*_{\uparrow\downarrow} & f^*_{\uparrow\uparrow} \end{pmatrix} \in \mathbb{H}. \tag{109}$$

If $H\psi = E\psi$, then also $H\Theta\psi = E\Theta\psi$. The wave-functions ψ and $\Theta\psi$ are orthogonal. Thus all eigenstates are doubly degenerate (Kramers doublett).

$$\overline{f_{\alpha,m_1,\beta,m_2}} = 0, \tag{110}$$
$$\overline{f_{\alpha_1,m_1,\alpha_1',m_1'} f_{\alpha_2,m_2,\alpha_2',m_2'}} = \frac{1}{2n} M(\delta_{\alpha_1,\alpha_2}\delta_{\alpha_1',\alpha_2'}\epsilon_{m_1,m_2}\epsilon_{m_1',m_2'}$$
$$+ \delta_{\alpha_1,\alpha_2'}\delta_{\alpha_1',\alpha_2}\epsilon_{m_1,m_2'}\epsilon_{m_1',m_2}). \tag{111}$$

The level correlation is given by

$$C_{\mathrm{S}}(f) = 1 + \delta(f) - \frac{\sin^2(2\pi f)}{(2\pi f)^2} + \mathrm{Si}(2\pi f)\left(\frac{\cos(2\pi f)}{2\pi f} - \frac{\sin(2\pi f)}{(2\pi f)^2}\right)$$
$$= \delta(f) + \frac{(2\pi f)^4}{135} + O(f^6). \tag{112}$$

The self-repulsion is particularly strong in the symplectic case. The correlation increases proportional to f^4 for small f. Level correlations for the random matrix ensemble were calculated by Wigner,[41] Porter,[48] and Mehta[49,50] with different techniques.

2.5. *Hydrodynamics*

Diffusion For $q=0$ one obtains

$$\hat{\Pi}_0(z_j,z_i)=\frac{1}{N}\sum_{q'}\frac{1}{R_i^{\rm d}-z_i+\hat{t}_{q'}}\frac{1}{R_j^{\rm d}-z_j+\hat{t}_{q'}}=\frac{\hat{w}_0(R_j^{\rm d}-R_i^{\rm d})}{R_j^{\rm d}-z_j-R_i^{\rm d}+z_i}, \qquad (113)$$

$$\hat{w}_0-\hat{\Pi}_0(z_j,z_i)=\frac{\hat{w}_0(z_i-z_j)}{R_j^{\rm d}-z_j-R_i^{\rm d}+z_i} \qquad (114)$$

One obtains as $\omega\to 0$ for $\Im\omega>0$

$$\hat{w}_0-\hat{\Pi}_0(E+\frac{1}{2}\omega,E-\frac{1}{2}\omega)=\frac{-{\rm i}\omega\hat{w}_0}{2\pi\rho(E)}. \qquad (115)$$

Thus a diffusion pole appears in the hydrodynamic limit

$$\hat{\Gamma}_q(E+\frac{1}{2}\omega,E-\frac{1}{2}\omega)=\frac{1}{\hat{w}_q-\hat{\Pi}_q(E+\frac{1}{2}\omega,E-\frac{1}{2}\omega)}=\frac{2\pi\rho(E)/\hat{w}_0^2}{-{\rm i}\omega+D^{(0)}q^2}. \qquad (116)$$

where $\hat{w}_q$ and $\hat{\Pi}_q$ are expanded up to order q^2 to obtain $D^{(0)}$. $\hat{w}_q-\hat{\Pi}_q$ approaches zero as ω and q tend to zero. It is obtained by summing the bubbles Π,

$$\hat{w}_q\hat{\Gamma}_q=1+\hat{M}_q\hat{\Pi}_q+(\hat{M}_q\hat{\Pi}_q)^2+(\hat{M}_q\hat{\Pi}_q)^3+..., \qquad (117)$$

which describes multiple scattering of a particle-hole pair by disorder. It is called Diffuson.

Conductivity One obtains for the frequency dependent conductivity at temperature T from Kubo[51] and Greenwood[52]

$$\sigma_T(\omega)=\frac{1}{\omega}\int{\rm d}E(f_T(E-\frac{1}{2}\omega)-f_I(E+\frac{1}{2}\omega))\sigma(\omega,E) \qquad (118)$$

with the Fermi-function $f_T(E)=1/({\rm e}^{E/(k_{\rm B}T)}+1)$ and Boltzmann constant $k_{\rm B}$.

In the limit $T=0$ and $\omega=0$ one obtains the d.c. conductivity $\sigma(0,\mu)$ for the chemical potential μ. Evaluation of this limit yields

$$\sigma(0,\mu)=e^2\rho(\mu)D^{(0)}(\mu) \qquad (119)$$

with the elementary electric charge e.

The action of the nonlinear sigma-model is given in the hydrodynamic limit by

$$\mathcal{S}(Q)=\frac{1}{2}\pi\rho^{(0)}\sum_r{\rm str}(-D^{(0)}(\nabla Q(r))^2-2{\rm i}\omega\Lambda Q(r)). \qquad (120)$$

Note that $-(\nabla Q(r))^2$ is positive semi-definite, if expressed by the gradients of θ_i and ϕ_i in Eq. (72).

Orthogonal and Symplectic Case

In these time-reversal invariant cases one requires for the spinless (orthogonal) case real matrix elements $f_{r,r'}=f_{r',r}\in\mathbb{R}$ as in (105), $t_{r,r'}=t_{r',r}\in\mathbb{R}$. The matrix-elements f and t of the spin dependent (symplectic) case are quaternions, compare Eq. 109.

Around Eq. (117) we realized that in the unitary case two particles called Diffusons running in the opposite direction lead to the long-range behaviour both in time and space (small ω and small q). In the time-invariant case one finds besides these diffusons similar correlations for two particles running in the same direction, called Cooperons, as described by Vollhardt and Wölfle.[53,54] The two-particle Green's function reads now

$$\begin{aligned}\hat{G}_2(q_1,q_1',z_1;q_2,q_2',z_2) = {} & \delta_{q_1,q_1'}\hat{G}_1(q_1,z_1)\delta_{q_2,q_2'}\hat{G}_1(q_2,z_2) \\ & + \frac{1}{N}\delta_{q_1+q_2-q_1'-q_2'}\hat{G}^{(0)}(q_1,z_1)\hat{G}^{(0)}(q_1',z_1) \\ & \times \hat{G}^{(0)}(q_2,z_2)\hat{G}^{(0)}(q_2',z_2)(\hat{\Gamma}_{q_1-q_2'}(z_1,z_2)+\hat{\Gamma}_{q_1+q_2}(z_1,z_2))\end{aligned} \tag{121}$$

with the Cooperon in the second term $\hat{\Gamma}$. This contribution was called Cooperon by Altshuler *et al.*[55,56] in their papers on magnetoresistance, since magnetic effects destroy the Cooperon.

Symmetry relations There are symmetry relations between the ensembles for various symmetries.

$$\mathcal{S}(Q) = \frac{\kappa}{t}\int \mathrm{d}^d r\, \mathrm{str}[\frac{1}{2}(\nabla Q(r))^2 + \mathrm{i}\frac{\omega}{D^{(0)}}\Lambda Q(r)]. \tag{122}$$

with

$$\begin{array}{lll} \text{ensemble} & \kappa & 1/t \\ \text{unitary} & -1 & \frac{1}{2}\pi\rho^{(0)}D^{(0)} \\ \text{unitary} & +1 & \frac{1}{2}\pi\rho^{(0)}D^{(0)} \\ \text{orthogonal} & -1 & \frac{1}{4}\pi\rho^{(0)}D^{(0)} \\ \text{symplectic} & +1 & \frac{1}{2}\pi\rho^{(0)}D^{(0)} \end{array} \tag{123}$$

In a formal power expansion in t one obtains the relations

$$Z_{\mathrm{O}}(t) = Z_{\mathrm{Sp}}(-t), \quad Z_{\mathrm{U}}(t) = Z_{\mathrm{U}}(-t), \tag{124}$$

where one exchanges $\lambda_1 - 1$ and $1 - \lambda_2$ in an expansion in these quantities. These symmetries are not valid in general, but only for small t, where the missing contributions for $\lambda_2 < -1$ can be neglected. Corresponding relations for expectation values were found by Jüngling and Oppermann.[57,58]

There are two types of compact supergroups, the unitary one and the unitary-orthosymplectic one. Their elements U have the property $U^\dagger U = \mathbf{1}$ in the unitary case and a similar relation for the unitary-orthosymplectic case, where however the adjoint is differently defined, similarly to the timereversal operation. If one takes away all Grassmann variables, then only the boson-boson and the fermion-fermion block are left. In the unitary case both blocks are unitary matrices, in the unitary-orthosymplectic case one is orthogonal, the other one is unitary-symplectic. We have switched the boson-boson block to the pseudo-form. Instead one can express the action by the fermion-fermion block as Efetov et al.[22] did. These two possibilities yield the connections Eqs. (124).

β-function and conductance The corresponding β-functions for t read[60–64]

$$\tilde{\beta}(t) = \begin{cases} \epsilon t - t^2 - \frac{3}{4}\zeta(3)t^5 + \frac{27}{64}\zeta(4)t^6 + O(t^7) & \text{orthogonal ensemble} \\ \epsilon t - \frac{1}{2}t^3 - \frac{3}{8}t^5 + O(t^7) & \text{unitary ensemble} \\ \epsilon t + t^2 - \frac{3}{4}\zeta(3)t^5 - \frac{27}{64}\zeta(4)t^6 + O(t^7) & \text{symplectic ensemble} \end{cases} \tag{125}$$

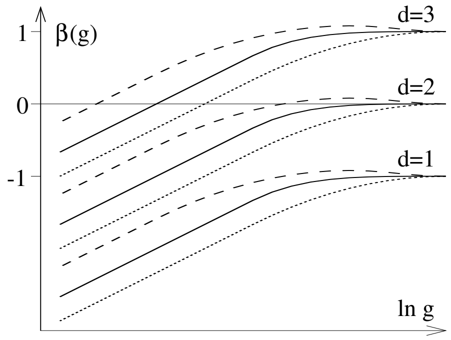

Fig. 1. Function $\beta(g)$ for dimensions $d = 1, 2, 3$ for the symplectic (upper dashed), unitary (middle full), and orthogonal (lower dotted line)

The conductance g (inverse resistance) of a cylindric body of length L and cross-section L^{d-1} is

$$g = \sigma L^{\epsilon} \propto t^{-1}. \tag{126}$$

The conductance was introduced by Abrahams et al.[24] Its variation as function of L is given by

$$\begin{aligned} \frac{\mathrm{d}\ln g}{\mathrm{d}\ln L} &= \beta(g) = t^{-1}\tilde{\beta}(t) = g\tilde{\beta}(1/g) \\ &= \begin{cases} \epsilon - g^{-1} - \frac{3}{4}\zeta(3)g^{-4} + \frac{27}{64}\zeta(4)g^{-5} + O(g^{-6}) & \text{orthogonal ensemble} \\ \epsilon - \frac{1}{2}g^{-2} - \frac{3}{8}g^{-4} + O(g^{-6}) & \text{unitary ensemble} \\ \epsilon + g^{-1} - \frac{3}{4}\zeta(3)g^{-4} - \frac{27}{64}\zeta(t)g^{-5} + O(g^{-6}) & \text{symplectic ensemble} \end{cases} \end{aligned} \tag{127}$$

For one-dimensional wires $\beta(g)$ is always negative. Therefore g decays with increasing length, as long as inelastic scattering is negligible.

For two-dimensional films β approaches 0 as g tends to infinity. Then the conductance decays slowly for the orthogonal and unitary case, although finally it will decay. This is the region of weak localization. The conductance increases in the symplectic case. This is called weak anti-localization and appears when spin-orbit interactions become important. It was predicted by Hikami et al.[65]

Three dimensional bodies with sufficiently strong conductance at a microscopical scale will become typical conductors with $g \propto L$. The mobility edge is obtained, when $\beta(g) = 0$. Then the conductance does not vary with length L. If g starts out with negative $\beta(g)$, then the body is an insulator.

Density fluctuations and multifractality The amplitudes of the wave-functions show charateristic fluctuations. They are described by the inverse participation ratio P

$$P_q(E) = \overline{\sum_i |\psi_i(r)|^{2q}\delta(E-e_i)}/\rho(E) \tag{128}$$

in the region of the localized states and by the participation ratio

$$1/p_q(E) = \lim_{\eta\to 0} \overline{[\sum_i |\psi_i(r)|^2\delta_\eta(E-e_i)]^q}/\rho(E)^q \tag{129}$$

in the region of extended states. In the later region one uses the smeared δ-function

$$\delta_\eta(x) = \frac{\eta}{\pi(x^2+\eta^2)} = \frac{1}{2\pi\mathrm{i}}\left(\frac{1}{x-\mathrm{i}\eta} - \frac{1}{x+\mathrm{i}\eta}\right) \tag{130}$$

These quantities were introduced by Bell and Dean[66] and discussed by Thouless[67] and Wegner[68] for $q=2$. P_2 is realized by a localized state consisting of $1/P_2$ sites with $|\psi^2(r)| = P_2$, and p_2 by an extended state with Np_2 sites with $|\psi^2(r)| = 1/(Np_2)$.

As long as the localization length ξ is small in comparison to the size L of the system, one obtains

$$P_q(E) \propto \xi^{-\tau_q} \propto |E-E_\mathrm{c}|^{\tau_q\nu}, \quad p_q(E) \propto \xi^{\Delta_q} \propto |E-E_\mathrm{c}|^{-\Delta_q\nu}. \tag{131}$$

The exponents are related by

$$\tau_q = (d-1)q + \Delta_q. \tag{132}$$

Δ_q vanishes for $q=0$ and $q=1$,

$$\tau_0 = d, \quad \Delta_0 = 0, \quad \tau_1 = 0, \quad \Delta_1 = 0. \tag{133}$$

Δ obeys the symmetry relation

$$\Delta_q = \Delta_{1-q}. \tag{134}$$

This has been observed in $2+\epsilon$-expansions.[59,61,62] General results were derived by Mirlin et al.[69] and by Gruzberg et al.[70,71]

Statistics of energy spacing The statistics of energy spacing is in the region of localized states of Poisson type $\Delta N \propto N$,
in the region of extended states of Wigner-Dyson type $\Delta N \propto \ln N$,
and at the mobility edge of type $\Delta N \propto N^{1-1/(d\nu)}$ as found by Kravtsov et al.[72]

Final remarks This article cannot cover all the results on the Anderson transition. We mention only three other important developments.

Until now we considered only the three classes introduced by Dyson, called Wigner-Dyson classes. Altland and Zirnbauer[73] found that all compact symmetric spaces correspond to a class of nonlinear sigma-models. The seven classes besides the three Wigner-Dyson classes are known as three chiral classes und four Bogolubov-de Gennes classes. They differ in the statistics of the states close to the band center from the Wigner-Dyson classes. They are classified according to their behaviour under time-reversal invariance, particle hole symmetry and sublattice symmetry. The reader is referred to the papers by Zirnbauer,[31,33] the review by Evers aand Mirlin,[36] and the article by Chiu et al.[74] This leads also into the field of topological insulators and superconductors.

Fröhlich and Spencer[75] proved that there is no diffusion in the Anderson tight-binding model for sufficiently large disorder or at low energies.

Kramer, MacKinnon et al.[76,77] estimated critical exponents numerically by finite size scaling. They obtained for the exponent ν

$$\begin{array}{rl} 1.57 \pm 0.02 & d = 3 \text{ orthogonal symmetry} \\ 1.43 \pm 0.04 & d = 3 \text{ unitary symmetry} \\ 1.375 \pm 0.016 & d = 3 \text{ symplectic symmetry} \\ 2.73 \pm 0.02 & d = 2 \text{ symplectic symmetry} \\ 2.593 \pm 0.006 & \text{integer quantum Hall effect} \end{array} \tag{135}$$

Here we considered only systems with one-particle hamiltonians. When interaction between the particles, in particular the Coulomb interaction between electrons has to be considered, then the supersymmetry approach does no longer help and one has to return to the replica trick or to diagramatic methods. This has been done among others by Altshuler, Aronov and Lee,[78] by Finkel'stein,[79,80] and by Belitz and Kirkpatrick.[30]

Note added in proof Recently Martin Zirnbauer has published two papers,[81,82] in which he argues for a pure-point region between the absolute continuous metallic and the singular continuous insulating region. Indications for a different behaviour of the transition is given by the paper by Garcia-Mata[84] et al. focusing on the insulator side of the transition finding a Kosterlitz-Thouless type of transition and and by the paper by Sierant[83] et al. approaching the transition from the metallic side finding a non-Kosterlitz thouless transition. These renormalization group calculations are governed by two couplings. The suggested renormalization group flows are depicted in Fig. 1 in[81] and Figures 1 and 3 in.[82] This is in contrast to the $2 + \epsilon$ expansion with only one coupling, which does not show a pure-point region.

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